SURFACES OBTAINED FROM INVOLUTIONS GENERATED BY HOMOGRAPHIES OF PERIODS THREE, FIVE, AND THIRTEEN

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CHAPTER I

INTRODUCTION

The equations in homogeneous coordinates of the extended complex plane

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^{\alpha}x_3$$

define a plane cyclic homography[1] of period p, where p is a prime number greater than two, E is a primitive pth root of unity, and α is an integer greater than unity and less than p. This homography generates an involution I_p of period p. We shall define a group of this involution as the following p points:

$$(x_1,x_2,x_3),\ (x_1,Ex_2,E^{\alpha}x_3),\ (x_1,E^2x_2,E^{2\alpha}x_3),\ \ldots,$$

$$(x_1,E^{p-1}x_2,E^{(p-1)\alpha}x_3).$$

In the latter (p-2) points of this group the third coordinate may be simplified whenever the exponent of E in this coordinate is equal to or greater than p.

The involution I_p possesses three points of coincidence. These points, which are the vertices of the triangle of reference, are $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$.

There are p linear systems of plane curves of order p

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[1] For a definition see Lucien Godeaux, Geometrie Algebrique (Liege: Soiences et Lettres, 1948), Tome I, pp. 134, 139.

which are invariant under H. They are given by the equations

(1) $\Sigma c_{ijk} x_1^i x_2^j x_3^k = 0, \quad i + j + k = p, \; j + \alpha k \equiv 0 \pmod{p},$

(2) $\Sigma c_{ijk} x_1^i x_2^j x_3^k = 0, \quad i + j + k = p, \; j + \alpha k \equiv 1 \pmod{p},$

.....

(p) $\Sigma c_{ijk} x_1^i x_2^j x_3^k = 0, \quad i + j + k = p, \; j + \alpha k \equiv p - 1 \pmod{p}.$

Only system (1) has no base points. The others have for base points O_1, O_2, and O_3 with fixed tangents at these points.

Let the number of terms in equation (1) be represented by (r + 1). The first (r - 1) terms can be arranged in descending powers of x_1, the rth term may contain only the variable x_2^p, and the (r + 1)th term only the variable x_3^p. The powers of x_1 in this equation may be arranged as follows:

$$p = i_1 > i_2 > \ldots > i_{r-1}.$$

Now, we define n(i) as a function of i by taking

$$n(i_1) = 1,$$

$$n(i_{m+1}) = n(i_m) + 1.$$

For n(i) the domain of the independent variable will be the values $i_m (m = 1, 2, \ldots, r - 1)$. Hence, we may relate projectively the curves of linear system (1) to the hyperplanes

of a space S_r of r dimensions by taking

$$\rho x_{n(i)} = x_1^i x_2^j x_3^k,$$

(T)
$$\rho x_r = x_2^p,$$

$$\rho x_{r+1} = x_3^p,$$

where

$$i + j + k = p,$$
$$j + \alpha k \equiv 0 \pmod{p}.$$

A surface ϕ in S_r is called the image of I_p if a point of ϕ corresponds to a group of I_p, and inversely. We may obtain the equations of this surface ϕ by eliminating ρ and $x_m (m = 1,2,3)$ in T. By solving the first, rth, and (r + 1)th equations of T for x_1, x_2, and x_3 respectively, we get

$$x_1 = \sqrt[p]{\rho x_1},$$

$$x_2 = \sqrt[p]{\rho x_r},$$

$$x_3 = \sqrt[p]{\rho x_{r+1}}.$$

Substituting these values in the remaining equations of T we obtain

$$\rho x_{n(i)} = \sqrt[p]{\rho^i x_1^i \, \rho^j x_r^j \, \rho^k x_{r+1}^k},$$

where i takes on the values $i_m (m = 2,3, \ldots , r - 1)$. By raising both members of these equations to the pth power and simplifying, we get

$$X^p_{n(i)} = X^i_1 X^j_r X^k_{r+1},$$

where

$$i = i_m (m = 2,3, \ldots , r - 1),$$

$$i + j + k = p,$$

$$j + \alpha k \equiv 0 \pmod{p}.$$

It is obvious that the coordinates of any point of ϕ will satisfy the above $(r - 2)$ equations; but, an example[2] will show that the inverse is not true. Hence, we conclude that the above $(r - 2)$ hypersurfaces may have not only in common surface ϕ but also another surface which is not in general an image of I_p.

By T point O_1 transforms into $O'_1(1,0, \ldots ,0)$, O_2 into $O'_r(0, \ldots , 0,1,0)$, and O_3 into $O'_{r+1}(0, \ldots ,0,1)$. Since O'_1, O'_r, and O'_{r+1} lie on each of the above $(r - 2)$ hypersurfaces, we see that these points also lie on the surface ϕ. They could not lie only on the other surface determined by these hypersurfaces because they are related projectively by T to points of the plane.

Finally, we shall show that the surface ϕ is of

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[2] See Appendix I.

order p. Since the linear system (1) has no base points, two curves of this system intersect in p^2 points which will depend only on the coefficients of the two curves considered. These points form p groups of I_p. Therefore, the two hyperplanes related to these two curves by T will intersect in p points which lie on ϕ. Hence, by the definition[3] of the order of a surface, we see that the above conclusion is reached.

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[3]For this definition see F. S. Woods, Higher Geometry (Boston: Ginn and Company, 1922), p. 390.

CHAPTER II

A SURFACE OBTAINED FROM AN INVOLUTION OF PERIOD THREE

Consider the homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3,$$

where E is a primitive cube root of unity. This homography generates an involution I_3 of period three. The following three points constitute a group of this involution:

$$(x_1,x_2,x_3),\ (x_1,Ex_2,E^2x_3),\ (x_1,E^2x_2,Ex_3).$$

The plane curves which are invariant under H form three linear systems. These systems have the equations

$$(1) \qquad v_1x_1^3 + v_2x_1x_2x_3 + v_3x_2^3 + v_4x_3^3 = 0,$$

$$(2) \qquad u_1x_1^2x_2 + u_2x_1x_3^2 + u_3x_2^2x_3 = 0,$$

$$(3) \qquad u_1x_1^2x_3 + u_2x_1x_2^2 + u_3x_2x_3^2 = 0.$$

The curves of linear system (1) are related projectively to the planes of S_3 by the following transformation:

$$(T) \qquad \frac{X_1}{x_1^3} = \frac{X_2}{x_1x_2x_3} = \frac{X_3}{x_2^3} = \frac{X_4}{x_3^3}.$$

By eliminating $x_m (m = 1,2,3)$ in T we obtain a surface ϕ, which is the image of I_3, given by the equation

$$(\phi) \qquad x_2^3 = x_1 x_3 x_4.$$

This surface is of order three.

To a plane curve of order three which is not invariant under H corresponds on ϕ a curve of order nine. This curve is cut out on ϕ by a cubic surface. Furthermore, the coefficients of the equation of the latter surface are functions of the coefficients of the equation of the plane curve considered.

In order to see this, consider the plane cubic curve

$$\Theta_1 \equiv \Sigma c_{ijk} x_1^i x_2^j x_3^k = 0,$$

where

$$i + j + k = 3.$$

Applying H twice in succession to Θ_1 we get

$$\Theta_2 \equiv \Sigma E^j + 2k c_{ijk} x_1^i x_2^j x_3^k = 0,$$

$$\Theta_3 \equiv \Sigma E^{2j} + k c_{ijk} x_1^i x_2^j x_3^k = 0.$$

The curve

$$(4) \qquad \Theta_1 \Theta_2 \Theta_3 = 0$$

corresponds on ϕ to a curve C in birational correspondence with each of the cubics $\Theta_m = 0 (m = 1,2,3)$. Actually, to a point of C correspond three points of the plane with one of these three points on each of the three plane curves considered.

The curve (4) meets a curve of linear system (1) in nine groups of I_3. Therefore, the plane which is related to this curve of system (1) by T intersects C in nine points. Consequently, C is of order nine.

There exist on $\theta_1 = 0$ nine couples of points to which correspond nine points on C. We conclude from this that C possesses nine variable double points which are simple points on ϕ. The couples of points which are under consideration here constitute the eighteen points of intersection of $\theta_1 = 0$ with $\theta_2 = 0$ and $\theta_3 = 0$.

Let us vary $\theta_1 = 0$ in a continuous manner in its plane until its equation becomes equation (1). The corresponding curve C varies on ϕ and reduces to the section of ϕ by the plane

$$v_1X_1 + v_2X_2 + v_3X_3 + v_4X_4 = 0$$

counted three times. That is, the section of ϕ is made by the reducible cubic surface

$$(v_1X_1 + v_2X_2 + v_3X_3 + v_4X_4)^3 = 0.$$

We conclude from this fact that the curves C are cut out on ϕ by cubic surfaces.

Now, let us consider $\theta_1 = 0$ varying in its plane and becoming equation (3). The curve (4) becomes

$$(u_1x_1^2x_3 + u_2x_1x_2^2 + u_3x_2x_3^2)^3 = 0 \tag{5}$$

and the curve C a curve A counted three times. The cubic surface cutting out curve A on ϕ will osculate ϕ with three-point contact in each point of intersection, except in the domain[1] of certain singular points which will be discussed later. By simplifying equation (5) and applying T we get the following equation for this cubic surface:

$$\Psi(x_1,x_2,x_3,x_4) \equiv u_1^3x_1^2x_4 + 3u_1^2u_2x_1x_2^2 + 3u_1^2u_3x_1x_2x_4 +$$

$$3u_1u_2^2x_1x_2x_3 + 6u_1u_2u_3x_2^3 + 3u_1u_3^2x_2^2x_4 + u_2^3x_1x_3^2 + 3u_2^2u_3x_2^2x_3 +$$

$$3u_2u_3^2x_2x_3x_4 + u_3^3x_3x_4^2 = 0.$$

By T point O_1 projects into $O_1'(1,0,0,0)$, O_2 into $O_3'(0,0,1,0)$, and O_3 into $O_4'(0,0,0,1)$. Both the surfaces ϕ and $\Psi = 0$ pass through O_1', O_3', and O_4'. In the domain of each of these points $\Psi = 0$ touches simply the nappe of ϕ. This results from the fact that the curves A pass simply through each of the points O_1', O_3', and O_4'.[2]

Consider the surface F of order nine in S_4 whose equations are

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[1] For an explanation see Lucien Godeaux, Geometrie Algebrique (Liege: Sciences et Lettres, 1948), Tome I, pp. 24 ff.

[2] See Appendix II.

(F) $\qquad x_2^3 = x_1 x_3 x_4, \qquad\qquad x_5^3 = \Psi(x_1, x_2, x_3, x_4).$

Between ϕ and F we have a (1,3) correspondence. Now, we search for the branch points of ϕ for this correspondence. That is, what are the points of ϕ to which correspond three coincident points of F? The first condition these points must satisfy is $X_5 = 0$. Hence, $\Psi(x_1, x_2, x_3, x_4) = 0$. But we have seen that this surface osculates ϕ with three-point contact in each point of intersection except in the domain of O_1', O_3', and O_4'. In the domain of each of these points it touches simply one of the nappes of ϕ. Therefore, the correspondence will have branch points at these points of simple contact. Hence, we see there are three branch points and the surface F is consequently irreducible.

We shall demonstrate that the surface F is rational. Take the transformation

$$(T') \qquad \frac{X_1}{x_1^3} = \frac{X_2}{x_1 x_2 x_3} = \frac{X_3}{x_2^3} = \frac{X_4}{x_3^3} = \frac{X_5}{u_1 x_1^2 x_3 + u_2 x_1 x_2^2 + u_3 x_2 x_3^2} = \frac{1}{\rho}.$$

This transformation orders to each point of the plane a unique point of F. We shall show that the inverse is true. That is, to each point of F can also be ordered a unique point of the plane.

We obtain the first of the following equations from

T' and the remaining two by multiplying both members of the first by $x_1x_2x_3$ and applying T two times in succession:

$$u_1x_1^2x_3 + u_2x_1x_2^2 + u_3x_2x_3^2 = \rho x_5,$$

$$u_1x_1x_2x_3^2 + u_2x_3x_1^2x_3 + u_3x_4x_1x_2^2 = \rho x_2x_5,$$

$$u_1x_1x_4x_1x_2^2 + u_2x_1x_3x_2x_3^2 + u_3x_3x_4x_1^2x_3 = \rho x_2^2x_5.$$

Solving these for $x_1^2x_3$, $x_1x_2^2$, and $x_2x_3^2$ by Cramer's rule we obtain

$$x_1^2x_3 = \frac{\rho x_5\Delta_1}{\Delta},$$

$$x_1x_2^2 = \frac{\rho x_5\Delta_2}{\Delta},$$

$$x_2x_3^2 = \frac{\rho x_5\Delta_3}{\Delta},$$

where

$$\Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_2x_3 & u_3x_4 & u_1x_1 \\ u_3x_3x_4 & u_1x_1x_4 & u_2x_1x_3 \end{vmatrix}$$

and $\Delta_m (m = 1,2,3)$ are determinants obtained from Δ by replacing the mth column by 1, x_2, and x_2^2. Also, from T' we obtain

$$x_1^3 = \rho x_1,$$

$$x_2^3 = \rho x_3,$$

$$x_3^3 = \rho x_4.$$

Using these and the previous solutions we get

$$\frac{x_3}{x_1} = \frac{x_5\Delta_1}{x_1\Delta},$$

$$\frac{x_1}{x_2} = \frac{x_5\Delta_2}{x_3\Delta},$$

$$\frac{x_2}{x_3} = \frac{x_5\Delta_3}{x_4\Delta},$$

From the last two equations we obtain

$$(T'^{-1}) \qquad \frac{x_1}{x_5^2\Delta_2\Delta_3} = \frac{x_2}{x_3x_5\Delta_3\Delta} = \frac{x_3}{x_3x_4\Delta^2}.$$

Hence, we see that between the surface F and the plane there exists a birational correspondence and conclude that F is a rational surface.

The transformation

$$X_1' : X_2' : X_3' : X_4' : X_5' = X_1 : X_2 : X_3 : X_4 : EX_5,$$

which generates on F the involution that has ϕ for its image, corresponds to the involution in the plane generated by the homography

$$x_1' : x_2' : x_3' = x_1 : E^2x_2 : Ex_3.$$

This involution is different from the one generated by H, but both involutions have the same groups.

If the linear system (2) is used instead of the linear system (3), the results[3] are similar to those obtained in this chapter. However, the involution on the related rational surface F which has ϕ for its image corresponds to the involution in the plane generated by H.

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[3] For these results see Lucien Godeaux, "Etude elementaire sur l'homographie plane de periode trois et sur une surface cubique," Nouvelles Annales de Mathematiques, 4e Serie, Tome XVI (1916), pp. 49-61.

CHAPTER III

A SURFACE OBTAINED FROM AN INVOLUTION OF PERIOD FIVE

Consider the homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3 ,$$

where E is a primitive fifth root of unity. This homography generates an involution I_5 of period five. A group of I_5 is composed of the following five points:
(x_1,x_2,x_3), (x_1,Ex_2,E^2x_3), (x_1,E^2x_2,E^4x_3), (x_1,E^3x_2,Ex_3), (x_1,E^4x_2,E^3x_3).

The plane curves of the fifth order which are invariant under H form five linear systems. They are given by the equations

$$(1) \qquad v_1x_1^5 + v_2x_1^2x_2x_3^2 + v_3x_1x_2^3x_3 + v_4x_2^5 + v_5x_3^5 = 0,$$

$$(2) \qquad u_1x_1^4x_2 + u_2x_1^2x_3^3 + u_3x_1x_2^2x_3^2 + u_4x_2^4x_3 = 0,$$

$$(3) \qquad u_1x_1^4x_3 + u_2x_1^3x_2^2 + u_3x_1x_2x_3^3 + u_4x_2^3x_3^2 = 0,$$

$$(4) \qquad u_1x_1^3x_2x_3 + u_2x_1^2x_2^3 + u_3x_1x_3^4 + u_4x_2^2x_3^3 = 0,$$

$$(5) \qquad u_1x_1^3x_3^2 + u_2x_1^2x_2^2x_3 + u_3x_1x_2^4 + u_4x_2x_3^4 = 0.$$

We relate projectively the curves of linear system (1)

to the hyperplanes of S_4 by taking

$$(T) \qquad \frac{X_1}{x_1^5} = \frac{X_2}{x_1^2 x_2 x_3^2} = \frac{X_3}{x_1 x_2^3 x_3} = \frac{X_4}{x_2^5} = \frac{X_5}{x_3^5} .$$

By eliminating $x_m (m = 1,2,3)$ in T we obtain a surface ϕ given by the equations

$$(\phi) \qquad \begin{Vmatrix} X_1 X_5 & X_2 & X_3 \\ X_2^2 & X_3 & X_4 \end{Vmatrix} = 0.$$

This surface is the image of I_5 and is of order five.

To a plane curve of the fifth order which is not invariant under H corresponds on ϕ a curve of the twenty-fifth order. This curve is cut out on ϕ by a hypersurface of order five. Furthermore, the coefficients of the equation of this hypersurface are functions of the coefficients of the equation of the plane curve considered.

In order to see this, consider the plane curve of the fifth order

$$\Theta_1 \equiv \Sigma c_{ijk} x_1^i x_2^j x_3^k = 0,$$

where

$$i + j + k = 5.$$

Applying H four times in succession to the above equation we obtain

$$\theta_2 \equiv \Sigma E^{j+2k} c_{ijk} x_1^i x_2^j x_3^k = 0,$$

$$\theta_3 \equiv \Sigma E^{2j+4k} c_{ijk} x_1^i x_2^j x_3^k = 0,$$

$$\theta_4 \equiv \Sigma E^{3j+k} c_{ijk} x_1^i x_2^j x_3^k = 0,$$

$$\theta_5 \equiv \Sigma E^{4j+3k} c_{ijk} x_1^i x_2^j x_3^k = 0.$$

The curve

$$\theta_1\theta_2\theta_3\theta_4\theta_5 = 0 \tag{6}$$

corresponds to a curve C on ϕ, where C is in birational correspondence with each of the curves $\theta_m = 0$ (m=1,2,3,4,5). In effect, to a point of C correspond five points of the plane with one of these five points on each of the five curves considered.

The curve (6) meets a curve of linear system (1) in twenty-five groups of I_5. Therefore, the hyperplane related to this curve of system (1) by T meets curve C in twenty-five points and we conclude that C is of order twenty-five.

There exist on $\theta_1 = 0$ fifty couples of points to which correspond fifty points on C. One concludes from this that C possesses fifty variable double points which are simple points of ϕ. The couples of points which are under consideration here constitute the one hundred points

of intersection of $\theta_1 = 0$ with $\theta_m = 0$ $(m = 2,3,4,5)$.

Let us vary $\theta_1 = 0$ in a continuous manner in its plane until its equation becomes equation (1). The corresponding curve C varies on ϕ and reduces to the section of ϕ by the hyperplane

$$v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 + v_5x_5 = 0$$

counted five times. That is, the section of ϕ is made by the reducible hypersurface of the fifth order

$$(v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 + v_5x_5)^5 = 0.$$

We conclude from this fact that the curves C are cut out on ϕ by hypersurfaces of the fifth order.

Now, let us vary $\theta_1 = 0$ in its plane until its equation becomes equation (2). Equation (6) reduces to

$$(7) \qquad (u_1x_1^4x_2 + u_2x_1^2x_3^3 + u_3x_1x_2^2x_3^2 + u_4x_2^4x_3)^5 = 0.$$

The corresponding curve C reduces to a curve A counted five times. The hypersurface of order five cutting out A on ϕ will therefore osculate ϕ with five-point contact in each point of intersection, except at certain singular points which will be discussed later. By simplifying equation (7) and applying T we get the following equation for this hypersurface of order five:

$$\Psi(x_1,x_2,x_3,x_4,x_5) \equiv u_1^5x_1^4x_4 + 5u_1^4u_2x_1^3x_2x_3 + 5u_1^4u_3x_1^3x_3^2 +$$

$$5u_1^4u_4x_1^3x_3x_4 + 10u_1^3u_2^2x_1^3x_3x_5 + 20u_1^3u_2u_3x_1^3x_4x_5 + 20u_1^3u_2u_4x_1^2x_2^2x_4 +$$

$$10u_1^3u_3^2x_1^2x_2^2x_4 + 20u_1^3u_3u_4x_1^2x_3^3 + 10u_1^3u_4^2x_1^2x_2x_4^2 + 10u_1^2u_2^3x_1^2x_2^2x_5 +$$

$$30u_1^2u_2^2u_3x_1x_2^4 + 30u_1^2u_2^2u_4x_1^2x_2x_4x_5 + 30u_1^2u_2u_3^2x_1^2x_2x_4x_5 +$$

$$60u_1^2u_2u_3u_4x_1^2x_3x_4x_5 + 30u_1^2u_2u_4^2x_1^2x_4^2x_5 + 10u_1^2u_3^3x_1^2x_3x_4x_5 +$$

$$30u_1^2u_3^2u_4x_1^2x_4^2x_5 + 30u_1^2u_3u_4^2x_1x_2^2x_4^2 + 10u_1^2u_4^3x_1x_3^3x_4 + 5u_1u_2^4x_1^2x_2x_5^2 +$$

$$20u_1u_2^3u_3x_1^2x_3x_5^2 + 20u_1u_2^3u_4x_1^2x_4x_5^2 + 30u_1u_2^2u_3^2x_1^2x_4x_5^2 +$$

$$60u_1u_2^2u_3u_4x_1x_2^2x_4x_5 + 30u_1u_2^2u_4^2x_1x_3^2x_5 + 20u_1u_2u_3^3x_1x_2^2x_4x_5 +$$

$$60u_1u_2u_3^2u_4x_1x_3^2x_5 + 60u_1u_2u_3u_4^2x_1x_2x_4^2x_5 + 20u_1u_2u_4^3x_2^2x_4^2 +$$

$$5u_1u_3^4x_1x_3^2x_5 + 20u_1u_3^3u_4x_1x_2x_4^2x_5 + 30u_1u_3^2u_4^2x_1x_3x_4^2x_5 +$$

$$20u_1u_3u_4^3x_1x_4^3x_5 + 5u_1u_4^4x_2^2x_4^3 + u_2^5x_1^2x_5^3 + 5u_2^4u_3x_1x_2^2x_5^2 + 5u_2^4u_4x_2^4x_5 +$$

$$10u_2^3u_3^2x_2^4x_5 + 20u_2^3u_3u_4x_1x_2x_4x_5^2 + 10u_2^3u_4^2x_1x_3x_4x_5^2 + 10u_2^2u_3^3x_1x_2x_4x_5^2 +$$

$$30u_2^2u_3^2u_4x_1x_3x_4x_5^2 + 30u_2^2u_3u_4^2x_1x_4^2x_5^2 + 10u_2^2u_4^3x_2^2x_4^2x_5 +$$

$$5u_2u_3^4x_1x_3x_4x_5^2 + 20u_2u_3^3u_4x_1x_4^2x_5^2 + 30u_2u_3^2u_4^2x_2^2x_4^2x_5 +$$

$$20u_2u_3u_4^3x_3^3x_4x_5 + 5u_2u_4^4x_2x_4^3x_5 + u_3^5x_1x_4^2x_5^2 + 5u_3^4u_4x_2^2x_4^2x_5 +$$

$$10u_3^3u_4^2x_3^2x_4x_5 + 10u_3^2u_4^3x_2x_4^3x_5 + 5u_3u_4^4x_3x_4^3x_5 + u_4^5x_4^4x_5 = 0.$$

Using T we see that point O_1 is imaged by point $O_1'(1,0,0,0,0)$, O_2 by $O_4'(0,0,0,1,0)$, and O_3 by $O_5'(0,0,0,0,1)$. Both the surface ϕ and the hypersurface $\Psi = 0$ pass through O_1', O_4', and O_5'. In the domain of O_1' we have that $\Psi = 0$ touches simply the nappe of ϕ, in the domain of O_4' simply the nappe of ϕ, and in the domain of O_5' doubly the nappe of ϕ. This results from the fact that the image curves of linear system (2) are cut out on ϕ by $\Psi = 0$. And, these image curves pass simply through O_1', simply through O_4', and doubly through O_5'.[1]

Consider the surface F of order twenty-five in S_5 whose equations are

$$(F) \qquad \left\| \begin{matrix} x_1x_5 & x_2 & x_3 \\ x_2^2 & x_3 & x_4 \end{matrix} \right\| = 0, \qquad x_6^5 = \Psi(x_1,x_2,x_3,x_4,x_5)$$

Between ϕ and F there is a (1,5) correspondence.

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[1] J. Dessart, "Sur les surfaces representant l'involution engendree par une homographie de periode cinq du plan," Memoires de la Societe Royale des Sciences de Liege, 3e Serie, Tome XVI (1931), pp. 1-23.

We now seek the branch points for this correspondence. That is, we seek the points of ϕ to which correspond five coincident points of F. The first condition these points must satisfy is $X_6 = 0$. In other words, $\Psi(x_1, x_2, x_3, x_4, x_5) = 0$. But we have seen that this hypersurface osculates ϕ with five-point contact in each point of meeting except in the domain of O_1', O_4', and O_5', where it touches one of the nappes of ϕ. This correspondence will have branch points only at these points where the contact of $\Psi = 0$ with ϕ is less than five-point contact. Hence, if we count O_5' as a branch point twice, there will be four branch points for this correspondence. Consequently, the surface F is irreducible.

We shall demonstrate that the surface F is rational. Take the transformation

$$(T') \quad \frac{X_1}{x_1^5} = \frac{X_2}{x_1^2 x_2 x_3^2} = \frac{X_3}{x_1 x_2^2 x_3} = \frac{X_4}{x_2^5} = \frac{X_5}{x_3^5} =$$

$$\frac{X_6}{u_1 x_1^4 x_2 + u_2 x_1^2 x_3^3 + u_3 x_1 x_2^2 x_3^2 + u_4 x_2^4 x_3} = \frac{1}{\rho}.$$

This transformation orders to each point of the plane a unique point of F. We shall proceed to show that an inverse of T', which will order to each point of F a unique point of the plane, can be established.

We obtain the first of the following equations from

T' and the remaining three by multiplying both members of the first by $x_1^3 x_2^4 x_3^3$ and applying T three times in succession:

$$u_1 x_1^4 x_2 + u_2 x_1^2 x_3^3 + u_3 x_1 x_2^2 x_3^2 + u_4 x_2^4 x_3 = \rho x_6,$$

$$u_1 x_1 x_4 x_1^2 x_3^3 + u_2 x_1 x_5 x_2^4 x_3 + u_3 x_4 x_5 x_1^4 x_2 + u_4 x_2 x_4 x_1 x_2^2 x_3^2 = \rho x_2 x_3 x_6,$$

$$u_1 x_1^2 x_4 x_5 x_2^4 x_3 + u_2 x_1 x_2 x_4 x_5 x_1 x_2^2 x_3^2 + u_3 x_1 x_4^2 x_5 x_1^2 x_3^3 + u_4 x_2 x_4^2 x_5 x_1^4 x_2 = \rho x_2^2 x_3^2 x_6,$$

$$u_1 x_1^2 x_2 x_4^2 x_5 x_1 x_2^2 x_3^2 + u_2 x_1 x_2 x_4^2 x_5^2 x_1^4 x_2 + u_3 x_1^2 x_4^2 x_5^2 x_2^4 x_3 + u_4 x_1 x_2 x_4^3 x_5 x_1^2 x_3^3 = \rho x_2^3 x_3^3 x_6.$$

Solving these equations by Cramer's rule we have

$$x_1^4 x_2 = \frac{\rho x_6 \Delta_1}{\Delta},$$

$$x_1^2 x_3^3 = \frac{\rho x_6 \Delta_2}{\Delta},$$

$$x_1 x_2^2 x_3^2 = \frac{\rho x_6 \Delta_3}{\Delta},$$

$$x_2^4 x_3 = \frac{\rho x_6 \Delta_4}{\Delta},$$

where

$$\Delta = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ u_3x_4x_5 & u_1x_1x_4 & u_4x_2x_4 & u_2x_1x_5 \\ u_4x_2x_4^2x_5 & u_3x_1x_4^2x_5 & u_2x_1x_2x_4x_5 & u_1x_1^2x_4x_5 \\ u_2x_1x_2x_4^2x_5^2 & u_4x_1x_2x_4^3x_5 & u_1x_1^2x_2x_4^2x_5 & u_3x_1^2x_4^2x_5^2 \end{vmatrix}$$

and $\Delta_m (m = 1,2,3,4)$ are determinants obtained from Δ by replacing the mth column by 1, x_2x_3, $x_2^2x_3^2$, and $x_2^3x_3^3$. Also, we have from T' that

$$x_1^5 = \rho x_1,$$

$$x_1^2x_2x_3^2 = \rho x_2,$$

$$x_1x_2^3x_3 = \rho x_3,$$

$$x_2^5 = \rho x_4,$$

$$x_3^5 = \rho x_5.$$

Using these and the previous solutions we get

$$\frac{x_1}{x_2} = \frac{x_1\Delta}{x_6\Delta_1},$$

$$\frac{x_2}{x_3} = \frac{x_2\Delta}{x_6\Delta_2},$$

$$\frac{x_1}{x_2} = \frac{X_2\Delta}{X_6\Delta_3},$$

$$\frac{x_2}{x_3} = \frac{X_4\Delta}{X_6\Delta_4}.$$

It follows from the last two of these relations that

$$(T'^{-1}) \qquad \frac{x_1}{X_2X_4\Delta^2} = \frac{x_2}{X_4X_6\Delta_3\Delta} = \frac{x_3}{X_6^2\Delta_3\Delta_4}.$$

Therefore, between the surface F and the plane there exists a birational correspondence. Hence, we see that F is a rational surface.

The transformation

$$X_1' : X_2' : X_3' : X_4' : X_5' : X_6' = X_1 : X_2 : X_3 : X_4 : X_5 : EX_6,$$

which generates on F the involution that has ϕ for its image, corresponds to the involution in the plane generated by the homography H.

If the linear system (3), (4), or (5) is used instead of the linear system (2), one will get results[2] which will be similar to those obtained in this chapter. However, the

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[2]For the results obtained when linear system (5) is used see W. R. Hutcherson et M. A. Childress, "Etude d'une involution cyclique de periode cinq," Bulletin de l'Academie Royale de Belgique (Classe des Sciences), 6e Serie, Tome XL (1954), pp. 103-108.

involution on the related rational surface F which has ϕ for its image will correspond to an involution in the plane generated by a homography different from H. For the linear system (3) this homography will be

$$x_1' : x_2' : x_3' = x_1 : E^3x_2 : Ex_3,$$

for linear system (4)

$$x_1' : x_2' : x_3' = x_1 : E^2x_2 : E^4x_3,$$

and for linear system (5)

$$x_1' : x_2' : x_3' = x_1 : E^4x_2 : E^3x_3.$$

The involutions generated by these three homographies are different from the one generated by H, but all four involutions have the same groups.

CHAPTER IV

PROJECTIONS OF SURFACES OBTAINED FROM AN INVOLUTION OF PERIOD THIRTEEN

1. The Image Surface ϕ.

Consider the homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^4x_3,$$

where E is a primitive thirteenth root of unity. This homography generates an involution I_{13} of period thirteen. A group of I_{13} is composed of the following thirteen points: (x_1,x_2,x_3), (x_1,Ex_2,E^4x_3), (x_1,E^2x_2,E^8x_3), $(x_1,E^3x_2,E^{12}x_3)$, (x_1,E^4x_2,E^3x_3), (x_1,E^5x_2,E^7x_3), $(x_1,E^6x_2,E^{11}x_3)$, (x_1,E^7x_2,E^2x_3), (x_1,E^8x_2,E^6x_3), $(x_1,E^9x_2,E^{10}x_3)$, $(x_1,E^{10}x_2,Ex_3)$, $(x_1,E^{11}x_2,E^5x_3)$, $(x_1,E^{12}x_2,E^9x_3)$.

The complete linear system of order thirteen in the plane contains the following thirteen linear systems which are transformed into themselves by H:

$$(1) \qquad v_1x_1^{13} + v_2x_1^9x_2x_3^3 + v_3x_1^6x_2^5x_3^2 + v_4x_1^5x_2^2x_3^6 + v_5x_1^3x_2^9x_3 +$$

$$v_6x_1^2x_2^6x_3^5 + v_7x_1x_2^3x_3^9 + v_8x_2^{13} + v_9x_3^{13}=0,$$

(2) $u_1x_1^{12}x_2 + u_2x_1^8x_2^2x_3^3 + u_3x_1^5x_2^6x_3^2 + u_4x_1^4x_2^3x_3^6 + u_5x_1^3x_3^{10} +$

$u_6x_1^2x_2^{10}x_3 + u_7x_1x_2^7x_3^5 + u_8x_2^4x_3^9 = 0,$

(3) $u_1x_1^{11}x_2^2 + u_2x_1^7x_2^3x_3^3 + u_3x_1^6x_3^7 + u_4x_1^4x_2^7x_3^2 + u_5x_1^3x_2^4x_3^6 +$

$u_6x_1^2x_2x_3^{10} + u_7x_1x_2^{11}x_3 + u_8x_2^8x_3^5 = 0,$

(4) $u_1x_1^{10}x_2^3 + u_2x_1^9x_2^4 + u_3x_1^6x_2^4x_3^3 + u_4x_1^5x_2x_3^7 + u_5x_1^3x_2^8x_3^2 +$

$u_6x_1^2x_2^5x_3^6 + u_7x_1x_2^2x_3^{10} + u_8x_2^{12}x_3 = 0,$

(5) $u_1x_1^{12}x_3 + u_2x_1^6x_2^4 + u_3x_1^8x_2x_3^4 + u_4x_1^5x_2^5x_3^3 + u_5x_1^4x_2^2x_3^7 +$

$u_6x_1^2x_2^9x_3^2 + u_7x_1x_2^6x_3^6 + u_8x_2^3x_3^{10} = 0,$

(6) $u_1x_1^{11}x_2x_3 + u_2x_1^8x_2^5 + u_3x_1^7x_2^2x_3^4 + u_4x_1^4x_2^6x_3^3 + u_5x_1^3x_2^3x_3^7 +$

$u_6x_1^2x_3^{11} + u_7x_1x_2^{10}x_3^2 + u_8x_2^7x_3^6 = 0,$

(7) $u_1x_1^{10}x_2^2x_3 + u_2x_1^7x_2^6 + u_3x_1^6x_2^3x_3^4 + u_4x_1^5x_3^8 + u_5x_1^3x_2^7x_3^3 +$

$u_6x_1^2x_2^4x_3^7 + u_7x_1x_2x_3^{11} + u_8x_2^{11}x_3^2 = 0,$

(8) $u_1x_1^9x_2^3x_3 + u_2x_1^8x_3^5 + u_3x_1^6x_2^7 + u_4x_1^5x_2^4x_3^4 + u_5x_1^4x_2x_3^8 +$

$u_6x_1^2x_2^8x_3^3 + u_7x_1x_2^5x_3^7 + u_8x_2^2x_3^{11} = 0,$

(9) $u_1x_1^{11}x_3^2 + u_2x_1^8x_2^4x_3 + u_3x_1^7x_2x_3^5 + u_4x_1^5x_2^8 + u_5x_1^4x_2^5x_3^4 +$

$u_6x_1^3x_2^2x_3^8 + u_7x_1x_2^9x_3^3 + u_8x_2^6x_3^7 = 0,$

(10) $u_1x_1^{10}x_2x_3^2 + u_2x_1^7x_2^5x_3 + u_3x_1^6x_2^2x_3^5 + u_4x_1^4x_2^9 + u_5x_1^3x_2^6x_3^4 +$

$u_6x_1^2x_2^3x_3^8 + u_7x_1x_3^{12} + u_8x_2^{10}x_3^3 = 0,$

(11) $u_1x_1^9x_2^2x_3^2 + u_2x_1^6x_2^6x_3 + u_3x_1^5x_2^3x_3^5 + u_4x_1^4x_3^9 + u_5x_1^3x_2^{10} +$

$u_6x_1^2x_2^7x_3^4 + u_7x_1x_2^4x_3^8 + u_8x_2x_3^{12} = 0,$

(12) $u_1x_1^8x_2^3x_3^2 + u_2x_1^7x_3^6 + u_3x_1^5x_2^7x_3 + u_4x_1^4x_2^4x_3^5 + u_5x_1^3x_2x_3^9 +$

$u_6x_1^2x_2^{11} + u_7x_1x_2^8x_3^4 + u_8x_2^5x_3^8 = 0,$

(13) $u_1x_1^{10}x_3^3 + u_2x_1^7x_2^4x_3^2 + u_3x_1^6x_2x_3^6 + u_4x_1^4x_2^8x_3 + u_5x_1^3x_2^5x_3^5 +$

$u_6x_1^2x_2^2x_3^9 + u_7x_1x_2^{12} + u_8x_2^9x_3^4 = 0.$

We relate projectively the curves of linear system (1) to the hyperplanes of S_8 by taking

(T) $$\frac{x_1}{x_1^{13}} = \frac{x_2}{x_1^9x_2x_3^3} = \frac{x_3}{x_1^6x_2^5x_3^2} = \frac{x_4}{x_1^5x_2^2x_3^6} = \frac{x_5}{x_1^3x_2^9x_3} =$$

$$\frac{x_6}{x_1^2x_2^6x_3^5} = \frac{x_7}{x_1x_2^3x_3^9} = \frac{x_8}{x_2^{13}} = \frac{x_9}{x_3^{13}}.$$

By eliminating x_m (m = 1,2,3) in T we obtain a surface ϕ given by the equations[1]

$$(\phi) \qquad \left\| \begin{matrix} x_1x_3 & x_2x_6 & x_3 & x_4x_8 & x_5 & x_6 & x_7 \\ x_2^2 & x_4^2 & x_4 & x_6^2 & x_6 & x_7 & x_9 \end{matrix} \right\| = 0.$$

This surface is the image of $\mathbf{I}_{13}$ and is of the thirteenth order. We wish to study the singularity of ϕ at $O_1'(1,0,0,0,0,0,0,0,0,0)$.

2. Projection of Surface ϕ from O_1'.

The curves of linear system (1) which pass through O_1 are represented by equation (1) when we take $v_1 = 0$. We shall designate these curves by C_1 and the linear system which they form by $|C_1|$. The curves C_1 are related projectively to the hyperplanes of the space $X_1 = 0$. This projectivity gives in the space $X_1 = 0$ a surface ϕ_1, which is the projection of ϕ from O_1' on $X_1 = 0$. This surface ϕ_1 is given by the equations

$$(\phi_1) \qquad \left\| \begin{matrix} x_2x_6 & x_3 & x_4x_8 & x_5 & x_6 & x_7 \\ x_4^2 & x_4 & x_6^2 & x_6 & x_7 & x_9 \end{matrix} \right\| = 0.$$

To the curves C_1 correspond on ϕ_1 the sections made by the

[1] For other equations representing this surface see Appendix III.

hyperplanes of S_8 which pass through O_1'.

In order to study the singularity of the curves C_1 at O_1 we shall utilize the following quadratic transformations:

$$(I) \qquad x_1 : x_2 : x_3 = y_1^2 : y_1y_2 : y_2y_3,$$

$$(II) \qquad x_1 : x_2 : x_3 = y_1^2 : y_2y_3 : y_1y_3.$$

Transformation I makes the point $y_2 = y_3 = 0$ correspond to the point infinitely near O_1 on the line $x_3 = 0$. This point will be designated by O_{12}. Transformation II makes the point $y_2 = y_3 = 0$ correspond to the point infinitely near O_1 on the line $x_2 = 0$. We shall designate this point by O_{13}. Repeated applications of either of these transformations will be indicated by the second subscript of the symbol used to designate the point being repeated the same number of times.[2]

By applying transformation I three times in succession to the equation of curves C_1, we obtain

$$(14) \quad (v_2y_3^3 + v_3y_2y_3^2 + v_5y_2^2y_3 + v_8y_2^3)y_1^{39} +$$

$$(v_4y_3 + v_6y_2)y_1^{26}y_2^{10}y_3^5 + v_7y_1^{13}y_2^{20}y_3^9 + v_9y_2^{29}y_3^{13} = 0.$$

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[2] For a more detailed explanation see S. T. Gormsen, "Maps of Certain Algebraic Curves Invariant under Cyclic Involutions of Periods Three, Five, and Seven" (Unpublished form of Ph.D. dissertation, Dept. of Mathematics, University of Florida, 1953), pp. 1-3.

Now, applying the transformation II nine times in succession to the equation of curves C_1, we obtain

$$(15)\quad (v_2y_2 + v_9y_3)y_1^{117} + v_3y_1^{78}y_2^5y_3^{35} + v_4y_1^{104}y_2^2y_3^{12} +$$

$$v_5y_1^{39}y_2^9y_3^{70} + v_6y_1^{65}y_2^6y_3^{47} + v_7y_1^{91}y_2^3y_3^{24} + v_8y_2^{13}y_3^{105} = 0.$$

The curves C_1 have a 4-tuple point at O_1. From the previous equations we see that these curves have in common in the neighborhood of O_1:

(a) three triple points O_{12}, O_{122}, and O_{1222} infinitely near successively to O_1;

(b) nine simple points O_{13}, O_{133}, ... , and $O_{1333333333}$ infinitely near successively to O_1.

Hence, two curves of $|C_1|$ intersect $4^2 + 3\cdot3^2 + 9\cdot1^2$ or 52 times at O_1. Therefore, the system $|C_1|$ has degree[3] 169 - 52 or 117. Since two curves of $|C_1|$ intersect in 117 variable points and the curves C_1 are related projectively to the hyperplanes of the space $X_1 = 0$, we deduce that the surface ϕ_1 has order 117/13 or 9. Furthermore, since two curves of $|C_1|$ intersect 52 times at O_1, we also deduce that O_1' is multiple of order 52/13 or 4 for the surface ϕ.

From equation (14) and T we have

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[3]For a definition see Lucien Godeaux, Geometrie Algebrique (Liege: Sciences et Lettres, 1949), Tome II, p. 142.

$$\frac{X_2}{y_1^{39}y_3^3} = \frac{X_3}{y_1^{39}y_2y_3^2} = \frac{X_4}{y_1^{26}y_2^{10}y_3^6} = \frac{X_5}{y_1^{39}y_2^2y_3} = \frac{X_6}{y_1^{26}y_2^{11}y_3^5} =$$

$$\frac{X_7}{y_1^{13}y_2^{20}y_3^9} = \frac{X_8}{y_1^{39}y_2^3} = \frac{X_9}{y_2^{29}y_3^{13}}.$$

Now, let $y_3 = ky_2$. The above relations reduce to

$$\frac{X_2}{k^3y_1^{39}} = \frac{X_3}{k^2y_1^{39}} = \frac{X_4}{k^6y_1^{26}y_2^{13}} = \frac{X_5}{ky_1^{39}} = \frac{X_6}{k^5y_1^{26}y_2^{13}} =$$

$$\frac{X_7}{k^9y_1^{13}y_2^{26}} = \frac{X_8}{y_1^{39}} = \frac{X_9}{k^{13}y_2^{39}}.$$

By letting y_2 tend to zero we obtain the parametric equations

$$(16) \quad \frac{X_2}{k^3} = \frac{X_3}{k^2} = \frac{X_5}{k} = \frac{X_8}{1}, \; X_4 = X_6 = X_7 = X_9 = 0.$$

Hence, we see that the points of ϕ infinitely near O_1' and situated on the cubic cone (16) correspond to the points infinitely near O_{1222}.

Similarly, from equation (15) and T we obtain

$$\frac{X_2}{y_1^{117}y_2} = \frac{X_3}{y_1^{78}y_2^5y_3^{35}} = \frac{X_4}{y_1^{104}y_2^2y_3^{12}} = \frac{X_5}{y_1^{39}y_2^9y_3^{70}} = \frac{X_6}{y_1^{65}y_2^6y_3^{47}} =$$

$$\frac{X_7}{y_1^{91}y_2^3y_3^{24}} = \frac{X_8}{y_2^{13}y_3^{105}} = \frac{X_9}{y_1^{117}y_3}.$$

Again, placing $y_3 = ky_2$ we get

$$\frac{x_2}{y_1^{117}} = \frac{x_3}{k^{35}y_1^{78}y_2^{39}} = \frac{x_4}{k^{12}y_1^{104}y_2^{13}} = \frac{x_5}{k^{70}y_1^{39}y_2^{78}} =$$

$$\frac{x_6}{k^{47}y_1^{65}y_2^{52}} = \frac{x_7}{k^{24}y_1^{91}y_2^{26}} = \frac{x_8}{k^{105}y_2^{117}} = \frac{x_9}{ky_1^{117}} .$$

By letting y_2 tend to zero, we obtain the parametric equations

(17) $$\frac{x_2}{1} = \frac{x_9}{k}, \quad x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0.$$

Therefore, the points of ϕ infinitely near $0_1'$ and situated on the plane (17) correspond to the points infinitely near $0_{1333333333}$.

To the points of the plane infinitely near 0_{1222} correspond on ϕ_1 the points of the skew cubic

$$(\gamma_1) \qquad \left\| \begin{matrix} x_2 & x_3 & x_5 \\ x_3 & x_5 & x_8 \end{matrix} \right\| = 0, \; x_1 = x_4 = x_6 = x_7 = x_9 = 0.$$

This skew cubic is the section of the cone (16) by the hyperplane $x_1 = 0$. To the points of the plane infinitely near $0_{1333333333}$ correspond on ϕ_1 the points of the line

$$(\gamma_2) \qquad x_1 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0.$$

This line is the section of the plane (17) by the hyperplane

$x_1 = 0$.

The tangent cone to the surface ϕ at O_1' decomposes into the cone projecting the cubic γ_1 and the plane projecting the line γ_2 from this point. Also, this tangent plane is contained in ϕ.

The curve γ_1 and the line γ_2 have in common the point $O_2'(0,1,0,0,0,0,0,0,0,0)$.

3. Projection of Surface ϕ_1 from O_2'.

The curves of the system $|C_1|$ which have distinct tangent lines $x_2 = 0$ and $x_3 = 0$ at O_1 are represented by equation (1) when $v_1 = v_2 = 0$. We shall designate these curves by C_2 and the linear system which they form by $|C_2|$. There exists a projectivity between the curves C_2 and the hyperplanes of the space $X_1 = X_2 = 0$. In this projectivity a surface ϕ_2 corresponds to the points of the plane. The surface ϕ_2 is the projection of surface ϕ_1 from O_2' on the hyperplane $X_2 = 0$ of the space $X_1 = 0$. This surface ϕ_2 is given by the equations

$$(\phi_2) \qquad \left\| \begin{matrix} x_3 & x_4x_8 & x_5 & x_6 & x_7 \\ x_4 & x_8^2 & x_6 & x_7 & x_9 \end{matrix} \right\| = 0.$$

To the curves C_2 correspond on ϕ_2 the sections made by the hyperplanes of S_8 which pass through O_1' and O_2'.

By applying transformation I once to the equation of curves C_2 we obtain

(18) $v_3y_1^{17}y_3^2 + v_4y_1^{12}y_2y_3^6 + v_5y_1^{15}y_2^3y_3 + v_6y_1^{10}y_2^4y_3^5 +$

$v_7y_1^5y_2^5y_3^9 + v_8y_1^{13}y_2^6 + v_9y_2^6y_3^{13} = 0.$

Now, applying I two times in succession to equation (18) yields

(19) $(v_3y_3^2 + v_5y_2y_3 + v_8y_2^2)y_1^{39} + (v_4y_3 + v_6y_2)y_1^{26}y_2^9y_3^5 +$

$v_7y_1^{13}y_2^{19}y_3^9 + v_9y_2^{28}y_3^{13} = 0.$

The application of transformation II once to the equation of the curves C_2 gives

(20) $v_3y_1^{14}y_2^5 + v_4y_1^{16}y_2^2y_3 + v_5y_1^7y_2^9y_3^3 + v_6y_1^9y_2^6y_3^4 + v_7y_1^{11}y_2^3y_3^5 +$

$v_8y_2^{13}y_3^6 + v_9y_1^{13}y_3^6 = 0.$

Now, applying II twice in succession to equation (20) gives

(21) $v_3y_1^{22}y_2^5y_3^2 + v_4y_1^{27}y_2^2 + v_5y_1^{11}y_2^9y_3^9 + v_6y_1^{16}y_2^6y_3^7 +$

$v_7y_1^{21}y_2^3y_3^5 + v_8y_2^{13}y_3^{16} + v_9y_1^{26}y_3^3 = 0$

and

(22) $v_3y_1^{30}y_2^5y_3^5 + v_4y_1^{38}y_2^2 + v_5y_1^{15}y_2^9y_3^{16} + v_6y_1^{23}y_2^6y_3^{11} +$

$v_7y_1^{31}y_2^3y_3^6 + v_8y_2^{13}y_3^{27} + v_9y_1^{39}y_3 = 0.$

By applying I to equation (21) yields nothing; but, by applying it once to equation (22) yields

$$(23)\quad (v_4y_2 + v_9y_3)y_1^{65} + v_3y_1^{52}y_2^{9}y_3^{5} + v_5y_1^{26}y_2^{24}y_3^{16} +$$
$$v_6y_1^{39}y_2^{16}y_3^{11} + v_7y_1^{52}y_2^{8}y_3^{6} + v_8y_2^{39}y_3^{27} = 0.$$

Finally, by applying I twice in succession to equation (20) yields

$$(24)\quad (v_3y_2 + v_4y_3)y_1^{26} + v_5y_1^{13}y_2^{11}y_3^{3} + v_6y_1^{13}y_2^{10}y_3^{4} +$$
$$v_7y_1^{13}y_2^{9}y_3^{5} + v_8y_2^{21}y_3^{6} + v_9y_1^{13}y_2^{8}y_3^{6} = 0.$$

The curves C_2 have multiplicity seven at O_1. From the previous equations we see that these curves have in common in the neighborhood of O_1:

(a) three successive double points O_{12}, O_{122}, and O_{1222} infinitely near successively to O_1;

(b) one triple point O_{13} infinitely near O_1;

(c) one double point O_{133} infinitely near O_{13};

(d) two simple points O_{1333} and O_{13332} infinitely near successively to O_{133};

(e) two simple points O_{132} and O_{1322} infinitely near successively to O_{13}.

Hence, two curves of the system $|C_2|$ intersect $7^2 + 1\cdot3^2 + 4\cdot2^2 + 4\cdot1^2$ or 78 times at O_1. Therefore, the system $|C_2|$ has degree 169 - 78 or 91. Since two curves of $|C_2|$ intersect in 91 variable points and the curves C_2 are related projectively to the hyperplanes of the space $X_1 = X_2 = 0$,

we deduce that the surface ϕ_2 has order 91/13 or 7. Furthermore, since two curves of $|C_2|$ intersect 78 times at O_1, we deduce that the sum of the order of O_1' for surface ϕ and the order of O_2' for surface ϕ_1 is 78/13 or 6. But O_1' is multiple of order 4 for ϕ. Hence, O_2' is multiple of order 2 for ϕ_1.

By using the same method as was employed in the previous projectivity, we see that to the points of the plane infinitely near O_{1222} correspond on ϕ_2 the points of the conic

(γ_1') $\quad X_3X_8 = X_5^2, \quad X_1 = X_2 = X_4 = X_6 = X_7 = X_9 = 0.$

This conic is the projection of the skew cubic γ_1 from O_2'. Also, to the points of the plane infinitely near O_{13332} correspond on ϕ_2 the points of the line

(γ_3) $\quad X_1 = X_2 = X_3 = X_5 = X_6 = X_7 = X_8 = 0.$

Finally, to the points of the plane infinitely near O_{1322} correspond on ϕ_2 the points of the line

(γ_4) $\quad X_1 = X_2 = X_5 = X_6 = X_7 = X_8 = X_9 = 0.$

The tangent planes to the surface ϕ_1 at O_2' are the planes projecting the lines γ_3 and γ_4 from this point. Also, the plane projecting the line γ_3 from O_2' is contained in ϕ_1.

The conic γ_1' and the line γ_4 meet at the point $O_3'(0,0,1,0,0,0,0,0,0)$ and the line γ_3 intersects the line γ_4 at $O_4'(0,0,0,1,0,0,0,0,0)$. The conic γ_1' and the line

Y_3 do not meet.

4. Projection of Surface ϕ_2 from O_3'.

The curves of the system $|C_2|$ with distinct tangent lines $x_2 = 0$ and $x_3 = 0$ at O_1 are represented by equation (1) when $v_1 = v_2 = v_3 = 0$. We shall designate these curves by C_3 and the linear system which they form by $|C_3|$. There exists a projectivity between the curves C_3 and the hyperplanes of the space $X_1 = X_2 = X_3 = 0$. This projectivity gives in this space a surface ϕ_3, which is the projection of ϕ_2 from O_3' on the hyperplane $X_3 = 0$ of the space $X_1 = X_2 = 0$. This surface ϕ_3 is given by the equations

$$(\phi_3) \qquad \left\| \begin{matrix} X_4X_8 & X_5 & X_6 & X_7 \\ X_6^2 & X_6 & X_7 & X_9 \end{matrix} \right\| = 0.$$

To the curves C_3 correspond on ϕ_3 the sections made by the hyperplanes of S_8 which pass through O_1', O_2', and O_3'.

The curves C_3 have multiplicity eight at O_1. By utilizing the transformations I and II in a manner similar to that used in the previous projections, we see that the curves of $|C_3|$ have in common in the neighborhood of O_1:

(a) a triple point O_{12} infinitely near O_1;

(b) two simple points O_{122} and O_{1222} infinitely near successively to O_{12};

(c) a double point O_{123} infinitely near O_{12};

(d) two simple points O_{1233} and O_{12332} infinitely near successively to O_{123};

(e) two double points O_{13} and O_{133} infinitely near successively to O_1;

(f) two simple points O_{1333} and O_{13332} infinitely near successively to O_{133}.

Therefore, two curves of $|C_3|$ intersect $8^2 + 1\cdot3^2 + 3\cdot2^2 + 6\cdot1^2$ or 91 times at O_1. We deduce from this that the system $|C_3|$ has degree 169 - 91 or 78, the surface ϕ_3 has order 78/13 or 6, and the point O_3' has order 91/13 - 4 - 2 or 1 for surface ϕ_2.

To the points of the plane infinitely near O_{1222} correspond on ϕ_3 the points of the line

(ν_1'') $\quad X_1 = X_2 = X_3 = X_4 = X_6 = X_7 = X_9 = 0.$

This line is the projection of the conic ν_1' from O_3'. To the points of the plane infinitely near O_{12332} correspond on ϕ_3 the points of the line

(ν_5) $\quad X_1 = X_2 = X_3 = X_6 = X_7 = X_8 = X_9 = 0.$

Finally, to the points of the plane infinitely near O_{13332} correspond on ϕ_3 the points of the line

(ν_3) $\quad X_1 = X_2 = X_3 = X_5 = X_6 = X_7 = X_8 = 0.$

Since this line already belongs to ϕ_2, this is its proper projection from O_3'.

The tangent plane to the surface ϕ_2 at O_3' is the

plane projecting the line γ_5 from this point.

The lines γ_1'' and γ_5 intersect at $O_5'(0,0,0,0,1,0,0,0,0)$ and the lines γ_5 and γ_3 intersect at O_4'. However, the lines γ_1'' and γ_3 do not intersect.

5. Projection of Surface ϕ_3 from O_4'.

We shall designate by C_4 the curves of the system $|C_3|$ which have distinct tangent lines $x_2 = 0$ and $x_3 = 0$ at O_1. These curves are represented by equation (1) when $v_1 = v_2 = v_3 = v_4 = 0$. The linear system formed by the curves C_4 will be designated by $|C_4|$. There exists a projectivity between the curves C_4 and the hyperplanes of the space $X_1 = X_2 = X_3 = X_4 = 0$. This projectivity gives in this space a surface ϕ_4, which is the projection of ϕ_3 from O_4' on the hyperplane $X_4 = 0$ of the space $X_1 = X_2 = X_3 = 0$. This surface ϕ_4 has the equations

$$(\phi_4) \qquad \begin{Vmatrix} X_5 & X_6 & X_7 \\ X_6 & X_7 & X_9 \end{Vmatrix} = 0.$$

To the curves C_4 correspond on ϕ_4 the sections made by the hyperplanes of S_6 passing through O_1', O_2', O_3', and O_4'.

The curves C_4 have multiplicity ten at O_1. In the neighborhood of this point they have in common:

(a) three simple points O_{12}, O_{122}, and O_{1222} infinitely near in succession to O_1;

(b) three triple points O_{13}, O_{132}, and O_{1322} infinitely near in succession to O_1.

Therefore, two curves of the system $|C_4|$ intersect $10^2 + 3\cdot3^2 + 3\cdot1^2$ or 130 times at O_1. We deduce from this that the system $|C_4|$ has degree 169 - 130 or 39, the surface ϕ_4 has order 39/13 or 3, and the point O_4' has order 130/13 - 4 - 2 - 1 or 3 for surface ϕ_3.

To the points of the plane infinitely near O_{1222} correspond on ϕ_4 the points of the line

(γ_1'') $\qquad x_1 = x_2 = x_3 = x_4 = x_6 = x_7 = x_9 = 0.$

Since this line already belongs to ϕ_3, this is its proper projection from O_4'. To the points of the plane infinitely near O_{1322} correspond on ϕ_4 the points of the skew cubic

$$(\gamma_4') \qquad \begin{vmatrix} x_5 & x_6 & x_7 \\ x_6 & x_7 & x_9 \end{vmatrix} = 0, \quad x_1 = x_2 = x_3 = x_4 = x_8 = 0.$$

The tangent cone to the surface ϕ_3 at O_4' is the cone projecting the cubic γ_4' from this point.

The line γ_1'' and the skew cubic γ_4' intersect at O_5'.

6. Projection of Surface ϕ_4 from O_5'.

We shall designate by C_5 the curves of the system $|C_4|$ which have distinct tangent lines $x_2 = 0$ and $x_3 = 0$ at O_1. The linear system formed by these curves will be designated

by $|C_5|$. The curves C_5 are represented by equation (1) when $v_1 = v_2 = v_3 = v_4 = v_5 = 0$. The projectivity which exists between the curves C_5 and the hyperplanes of the space $X_1 = X_2 = X_3 = X_4 = X_5 = 0$ gives in this space a surface ϕ_5. This surface is the projection of ϕ_4 from O_5' on the hyperplane $X_5 = 0$ of the space $X_1 = X_2 = X_3 = X_4 = 0$. This surface ϕ_5 is represented by the equation

$$(\phi_5) \qquad X_6X_9 = X_7^2 .$$

To the curves C_5 correspond on ϕ_5 the sections made by the hyperplanes of S_8 passing through O_1', O_2', O_3', O_4', and O_5'.

The curves C_5 have multiplicity eleven at O_1. In the neighborhood of this point they have in common:

(a) two double points O_{12} and O_{123} infinitely near in succession to O_1;

(b) two simple points O_{1233} and O_{12332} infinitely near in succession to O_{123};

(c) three double points O_{13}, O_{132}, and O_{1322} infinitely near in succession to O_1.

Therefore, two curves of the system $|C_5|$ intersect $11^2 + 5\cdot2^2 + 2\cdot1^2$ or 143 times at O_1. We deduce from this that the system $|C_5|$ has degree 169 - 143 or 26, the surface ϕ_5 has order 26/13 or 2, and the point O_5' has order 143/13 - 4 - 2 - 1 - 3 or 1 for surface ϕ_4. Of course we already know that ϕ_5 has order 2 from the equation which represents it.

To the points of the plane infinitely near O_{12332} correspond on ϕ_5 the points of the line

(γ_5') $\quad x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = x_9 = 0.$

To the points of the plane infinitely near O_{1322} correspond on ϕ_5 the points of the conic

(γ_4'') $\quad x_6x_9 = x_7^2, \quad x_1 = x_2 = x_3 = x_4 = x_5 = x_8 = 0.$

This conic is the projection of the skew cubic γ_4' from O_5'.

The tangent plane to the surface ϕ_4 at O_5' is the plane projecting the line γ_5' from this point.

The line γ_5' and the conic γ_4'' intersect at $O_6'(0,0,0,0,0,1,0,0,0)$. From the equation representing the surface ϕ_5 we see that the tangent plane to this surface at O_6' has the equations

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_9 = 0.$$

7. Remarks.

We find that the neighborhood of the point O_{1322} in the plane is represented by the line γ_4 on the surface ϕ_2 and by the skew cubic γ_4' on the surface ϕ_4. Hence, it is necessary that the skew cubic γ_4' correspond to the line γ_4 in the projections that we have made. In the passage from ϕ_2 to ϕ_3 we projected from O_3', and under this projection the line γ_4 is projected into the point O_4'. Therefore, the points of the line γ_4 are represented projectively on ϕ_3 by the neighborhood of O_4'. Now, the skew

cubic ν_4' also represents projectively the neighborhood of O_4' on ϕ_3. We have shown previously that O_4' is triple for ϕ_3.

The point O_4' is simple for the surface ϕ_2. In order to see this, let us consider the curves C_2 when $v_4 = 0$ in the equation representing them. We shall designate these curves by C_2' and the linear system which they form by $|C_2'|$. The curves C_2' are represented by equation (1) when $v_1 = v_2 = v_4 = 0$. These curves have multiplicity seven at O_1, which is the same as for the curves C_2. In the neighborhood of O_1 the curves C_2' have in common:

(a) three double points O_{12}, O_{122}, and O_{1222} infinitely near successively to O_1;

(b) one 5-tuple point O_{13} infinitely near O_1;

(c) five simple points O_{133}, O_{1332}, O_{13322}, O_{133222}, and $O_{1332222}$ infinitely near successively to O_{13}.

Therefore, two curves of the system C_2' intersect $7^2 + 1\cdot5^2 + 3\cdot2^2 + 5\cdot1^2$ or 91 times at O_1. This means that system $|C_2'|$ has degree 169 - 91 or 78. In the space $X_1 = X_2 = 0$ the sections of the surface ϕ_2 by the hyperplanes passing through O_4' correspond to the curves C_2'. These sections form a linear system of degree 78/13 or 6. Since ϕ_2 is of order 7, we conclude that O_4' is simple for this surface.

Also, we find that the neighborhood of the point O_{12332} in the plane is represented by the line ν_5 on the

surface ϕ_3 and by the line γ_5' on the surface ϕ_5. Again, it is necessary that in our projections we have line γ_5' corresponding to line γ_5. In the passage from ϕ_3 to ϕ_4 we projected ϕ_3 from O_4'. Under this projection the line γ_5 is projected into O_5'. Hence, the points of the line γ_5 are represented projectively on ϕ_4 by the neighborhood of O_5'. But, the line γ_5' also represents projectively the neighborhood of O_5' on ϕ_4. We have shown previously that O_5' is simple for ϕ_4.

The point O_5' is also simple for the surface ϕ_3. In order to show this, let us consider the curves C_3 when $v_5 = 0$ in the equation representing them. We shall designate these curves by C_3' and the linear system which they form by $|C_3'|$. The curves C_3' are represented by equation (1) when $v_1 = v_2 = v_3 = v_5 = 0$. These curves have multiplicity eight at O_1, which is the same as for the curves C_3. In the neighborhood of O_1 curves C_3' have in common:

(a) one 5-tuple point O_{12} infinitely near O_1;

(b) five simple points O_{123}, O_{1232}, O_{12322}, O_{123222}, and $O_{1232222}$ infinitely near successively to O_{12};

(c) two double points O_{13} and O_{133} infinitely near successively to O_1;

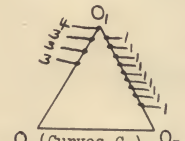
(d) two simple points O_{1333} and O_{13332} infinitely near successively to O_{133}.

Therefore, two curves of the system $|C_3'|$ intersect

$8^2 + 1\cdot5^2 + 2\cdot2^2 + 7\cdot1^2$ or 104 times at O_1. This implies the system $|C_3'|$ has degree 169 - 104 or 65. In the space $X_1 = X_2 = X_3 = 0$ the sections of the surface ϕ_3 by the hyperplanes passing through O_5' correspond to the curves C_3'. These sections form a linear system of degree 65/13 or 5. Since ϕ_3 is of order 6, we reach the conclusion that O_5' is simple for this surface.

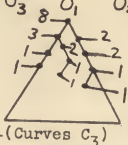
8. Summary.

The multiplicity of the curves C_m(m=1,2,3,4,5) at O_1 and at the points infinitely near O_1 in the invariant directions $x_2 = 0$ and $x_3 = 0$ has been stated previously. We may represent these points and their multiplicities for these curves by the following figures:

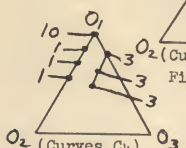


O_2(Curves C_1) Figure 1

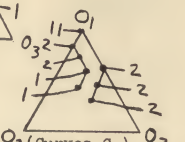
O_2(Curves C_2) Figure 2



O_2(Curves C_3) Figure 3



O_2(Curves C_4) Figure 4



O_2(Curves C_5) Figure 5

Some of the results regarding the surfaces ϕ and $\phi_m(m=1,2,3,4,5)$ obtained in this chapter may be listed as follows:

Surface	Order of Surface	Point on Surface	Multiplicity of Point for Surface	Tangent Elements to Surface at Point
ϕ	13	O_1'	4	$\begin{Vmatrix} x_2 & x_3 & x_5 \\ x_3 & x_5 & x_8 \end{Vmatrix} = 0$, $x_m = 0 (m=4,6,7,9)$ and $x_m = 0 (m=3,4,5,6,7,8)$
ϕ_1	9	O_2'	2	$x_m = 0 (m=1,3,5,6,7,8)$ and $x_m = 0 (m=1,5,6,7,8,9)$
ϕ_2	7	O_3'	1	$x_m = 0 (m=1,2,6,7,8,9)$
ϕ_3	6	O_4'	3	$\begin{Vmatrix} x_5 & x_6 & x_7 \\ x_6 & x_7 & x_9 \end{Vmatrix} = 0$, $x_m = 0 (m=1,2,3,8)$
ϕ_4	3	O_5'	1	$x_m = 0 (m=1,2,3,4,7,9)$
ϕ_5	2	O_6'	1	$x_m = 0 (m=1,2,3,4,5,9)$

A symbolic representation of the projections which we have made may be incorporated in a figure that follows. In this figure no attempt has been made to represent the surfaces ϕ and $\phi_m (m = 1,2,3,4,5)$.

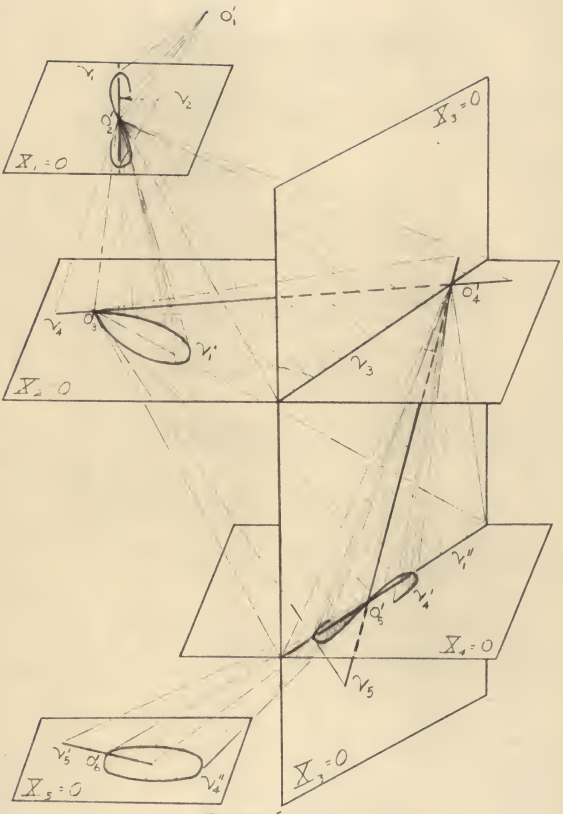

O'1
Γ1
Γ2
O'2
X1 = 0
X3 = 0
Γ4
O3
Γ1'
O4
X2 = 0
Γ3
Γ1''
Γ4'
O'5
X4 = 0
Γ5
Γ5'
O'6
Γ4''
X5 = 0
X6 = 0

Figure 6

Finally, let us remark that results similar to the preceding ones obtained in this chapter can be found by projecting the surface ϕ from O'_8 or O'_9. In order to see this for the point O'_8, we note that equation (1) remains unchanged when we apply the permutation $p_1 = (123)$ to the subscripts of x and the permutation $P_1 = (189)(257)(364)$ to the subscripts of v. Therefore, the results of projecting the surface ϕ from O'_8 and the additional results obtained thereafter by a series of successive projections similar to that performed previously may be obtained by applying the permutation P_1 to the subscripts of X in the previous results. In a similar manner, the application of the permutation $p_2 = (132)$ to the subscripts of x and the permutation $P_2 = (198)(275)(346)$ to the subscripts of v and X will yield the results from the previous ones for the series of successive projections starting with the projection of the surface ϕ from the point O'_9.

CHAPTER V

A SURFACE OBTAINED FROM AN INVOLUTION OF PERIOD THIRTEEN

We shall consider again the involution I_{13} of period thirteen from the preceding chapter. In this chapter H, T, ϕ, and equations (1) through (13) will refer to the homography H, the transformation T, the surface ϕ, and the equations (1) through (13) respectively of Chapter IV. Designations other than these will pertain only to this chapter.

Let us now consider the non-invariant plane curve of the thirteenth order

$$\theta_1 \equiv \Sigma a_{ijk} x_1^i x_2^j x_3^k = 0,$$

where

$$i + j + k = 13.$$

Applying H twelve times in succession to the above equation we obtain the twelve equations given by

$$\theta_n \equiv \Sigma \rho^{(n-1)j + w(n)\cdot k} a_{ijk} x_1^i x_2^j x_3^k = 0,$$

where

$$i + j + k = 13;$$

$$n = 2,3, \ldots , 13;$$

and $w(n)$ is the remainder when $4(n - 1)$ is divided by 13.

The curve

$$\theta_1 \theta_2 \ldots \theta_{13} = 0 \tag{14}$$

corresponds to a curve C on ϕ where C is in birational

correspondence with each of the curves $\Theta_m = 0$ $(m=1,2,\ldots,13)$. That is, to a point of C correspond thirteen points of the plane with one of these thirteen points on each of the thirteen curves considered.

The curve (14) meets a curve of linear system (1) in one hundred and sixty-nine groups of I_{13}. This implies that the hyperplane related to this curve of system (1) by T intersects curve C in one hundred and sixty-nine points. Hence, C is of order one hundred and sixty-nine. Later we shall show that the curve C is cut out on ϕ by a hypersurface of order thirteen and that the coefficients of the equation of this hypersurface are functions of the coefficients of the equation of the plane curve considered.

There exist on $\Theta_1 = 0$ one thousand and fourteen couples of points to which correspond one thousand and fourteen points on C. We conclude from this fact that the curve C possesses one thousand and fourteen double points which are simple points on ϕ. The couples of points which are under consideration here constitute the two thousand and twenty-eight points of intersection of $\Theta_1 = 0$ with $\Theta_m=0$ $(m=2,3,\ldots,13)$.

Let us vary $\Theta_1 = 0$ in a continuous manner in its plane until its equation becomes equation (1). The corresponding curve C varies on ϕ and reduces to the section of ϕ by the hyperplane

$$v_1X_1 + v_2X_2 + v_3X_3 + v_4X_4 + v_5X_5 + v_6X_6 + v_7X_7 + v_8X_8 + v_9X_9 = 0$$

counted thirteen times. This implies that the section of ϕ is made by the reducible hypersurface of the thirteenth order

$$(v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 + v_5x_5 + v_6x_6 + v_7x_7 + v_8x_8 + v_9x_9)^{13} = 0.$$

Hence, we reach the conclusion that the curves C are cut out on ϕ by hypersurfaces of the thirteenth order.

Suppose we now vary $\theta_1 = 0$ in its plane as before until its equation becomes equation (2). Equation (14) reduces to

$$(15) \qquad (u_1x_1^{12}x_2 + u_2x_1^8x_2^2x_3^3 + u_3x_1^5x_2^6x_3^2 + u_4x_1^4x_2^3x_3^6 + u_5x_1^3x_3^{10} + u_6x_1^2x_2^{10}x_3 + u_7x_1x_2^7x_3^5 + u_8x_2^4x_3^9)^{13} = 0.$$

The corresponding curve C reduces to a curve A counted thirteen times. The hypersurface of order thirteen cutting out curve A on ϕ will therefore osculate ϕ with thirteen-point contact in each point of intersection, except at certain singular points which will be discussed later. By simplifying equation (15) and applying T we obtain the following equation for this hypersurface of order thirteen:

$$\Psi(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9) \equiv u_1^{13}x_1^{12}x_8 + u_6^{13}x_1^2x_8^{10}x_9 + u_5^{13}x_1^3x_9^{10} + f(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9) = 0.$$

In this equation the expression $f(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)$

contains no term with an exponent of X_1 equal to or greater than twelve, no term with an exponent of X_8 equal to or greater than ten, and no term with an exponent of X_9 equal to or greater than ten. Also the constants in this expression will be functions of u_m $(m=1,2,\ldots,8)$.

Using T we see that O_1 is imaged by $O_1'(1,0,0,0,0,0,0,0,0)$, O_2 by $O_8'(0,0,0,0,0,0,0,1,0)$, and O_3 by $O_9'(0,0,0,0,0,0,0,0,1)$. Both the surface ϕ and the hypersurface $\Psi = 0$ pass through O_1', O_8', and O_9'. In the domain of O_1' we see that $\Psi = 0$ touches simply the nappe of ϕ, in the domain of O_8' triply the nappe of ϕ, and in the domain of O_9' simply the nappe of ϕ. We conclude this from the fact that the image curves of linear system (2) are cut out on ϕ by $\Psi = 0$. It can be shown that these image curves pass simply through O_1', triply through O_8', and simply through O_9'.[1]

Consider the surface F of order one hundred and sixty-nine in S_9 whose equations are

$$(F) \qquad \begin{Vmatrix} X_1X_3 & X_2X_6 & X_3 & X_4X_8 & X_5 & X_6 & X_7 \\ X_2^2 & X_4^2 & X_4 & X_6^2 & X_6 & X_7 & X_9 \end{Vmatrix} = 0,$$

$$X_{10}^{13} = \Psi(X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9).$$

Between ϕ and F we have a (1,13) correspondence. Now, we search for the branch points of ϕ for this correspondence. That is, we are searching for the points of ϕ to

[1] See Appendix III.

which correspond thirteen coincident points of F. The first condition that these points must satisfy is $X_{10} = 0$. For these points this implies that $\Psi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = 0$. But we have seen that this hypersurface osculates ϕ with thirteen-point contact in each point of meeting except in the domain of O_1', O_8', and O_9', where it touches one of the nappes of ϕ. There will be branch points only at these points where the contact of $\Psi = 0$ with ϕ is less than thirteen-point contact. Hence, if we count O_8' as a branch point three times, there will be five branch points for this correspondence. Consequently, the surface F is irreducible.

We shall demonstrate that the surface F is rational. Take the transformation

$$(T') \qquad \frac{X_1}{x_1^{13}} = \frac{X_2}{x_1^9 x_2 x_3^3} = \frac{X_3}{x_1^6 x_2^5 x_3^2} = \frac{X_4}{x_1^5 x_2^2 x_3^6} = \frac{X_5}{x_1^3 x_2^9 x_3} =$$

$$\frac{X_6}{x_1^2 x_2^6 x_3^5} = \frac{X_7}{x_1 x_2^3 x_3^9} = \frac{X_8}{x_2^{13}} = \frac{X_9}{x_3^{13}} = \frac{X_{10}}{g(x_1, x_2, x_3)} = \frac{1}{\rho},$$

where

$$g(x_1, x_2, x_3) \equiv u_1 x_1^{12} x_2 + u_2 x_1^8 x_2^2 x_3^3 + u_3 x_1^5 x_2^6 x_3^2 + u_4 x_1^4 x_2^3 x_3^6 +$$

$$u_5 x_1^3 x_3^{10} + u_6 x_1^2 x_2^{10} x_3 + u_7 x_1 x_2^7 x_3^5 + u_8 x_2^4 x_3^9 .$$

This transformation orders to each point of the plane a unique point of F. We shall now develop an inverse for T', which

will order to each point of F a unique point of the plane.

The first of the following equations is obtained from T'. The remaining seven are obtained by multiplying both members of the first by $x_1^{15}x_2^{19}x_3^{18}$ and applying T seven times in succession:

$$u_1x_1^{12}x_2 + u_2x_1^8x_2^2x_3^3 + u_3x_1^5x_2^6x_3^2 + u_4x_1^4x_2^3x_3^6 + u_5x_1^3x_3^{10} + u_6x_1^2x_2^{10}x_3 +$$
$$u_7x_1x_2^7x_3^5 + u_8x_2^4x_3^9 = \rho x_{10},$$

$$u_1x_1x_5x_6x_7x_1^8x_2^2x_3^3 + u_2x_1x_5x_6x_7x_1^4x_2^3x_3^6 + u_3x_1x_5x_6x_7x_1x_2^7x_3^5 +$$
$$u_4x_1x_5x_6x_7x_2^4x_3^9 + u_5x_5x_6x_7x_9x_1^{12}x_2 + u_6x_2x_5x_6x_8x_1^3x_3^{10} +$$
$$u_7x_2x_5x_6x_9x_1^2x_2^{10}x_3 + u_8x_2x_7x_8x_9x_1^5x_2^6x_3^2 = \rho x_2x_5x_6x_7x_{10},$$

$$u_1x_1^2x_5^2x_6^2x_7^2x_1^4x_2^3x_3^6 + u_2x_1^2x_5^2x_6^2x_7^2x_2^4x_3^9 + u_3x_1x_2x_5^2x_6^2x_7x_9x_1^2x_2^{10}x_3 +$$
$$u_4x_1x_2x_5x_6x_7^2x_8x_9x_1^5x_2^6x_3^2 + u_5x_1x_5^2x_6^2x_7^2x_9x_1^8x_2^2x_3^3 +$$
$$u_6x_2x_5^2x_6^2x_7x_8x_9x_1^{12}x_2 + u_7x_2^2x_5^2x_6^2x_8x_9x_1^3x_3^{10} +$$
$$u_8x_1x_2x_5x_6x_7^2x_8x_9x_1x_2^7x_3^5 = \rho x_2^2x_5^2x_6^2x_7x_{10},$$

$$u_1x_1^3x_5^3x_6^3x_7^2x_2^4x_3^9 + u_2x_1^2x_2x_5^2x_6^2x_7^3x_8x_9x_1^5x_2^6x_3^2 + u_3x_1x_2^2x_5^3x_6^3x_7x_8x_9x_1^3x_3^{10} +$$
$$u_4x_1^2x_2x_5^2x_6^2x_7^3x_8x_9x_1x_2^7x_3^5 + u_5x_1^2x_5^3x_6^3x_7^3x_9x_1^4x_2^3x_3^6 +$$
$$u_6x_1x_2x_5^3x_6^3x_7^3x_8x_9x_1^8x_2^2x_3^3 + u_7x_2^2x_5^3x_6^3x_7x_8x_9^2x_1^{12}x_2 +$$

$$u_8x_1x_2^2x_5^2x_6^2x_7^2x_8x_9^2x_1^2x_2^{10}x_3 = \rho x_2^3x_5^3x_6^3x_7^3x_{10},$$

$$u_1x_1^3x_2x_5^3x_6^3x_7^4x_8x_9x_1^5x_2^6x_3^2 + u_2x_1^3x_2x_5^3x_6^3x_7^4x_8x_9x_1^7x_2^5x_3^2 +$$

$$u_3x_1x_2^2x_5^4x_6^4x_7^2x_8x_9^2x_1^{12}x_2 + u_4x_1^2x_2^2x_5^3x_6^3x_7^3x_8x_9^2x_1^2x_2^{10}x_3 +$$

$$u_5x_1^3x_5^4x_6^4x_7x_9x_2^4x_3^9 + u_6x_1^2x_2x_5^4x_6^3x_7^3x_8x_9x_1^4x_2^3x_3^6 +$$

$$u_7x_1x_2^2x_5^4x_6^4x_7^2x_8x_9^2x_1^8x_2^2x_3^3 + u_8x_1x_2^3x_5^3x_6^3x_7^2x_8^2x_9^2x_1^3x_3^{10} =$$

$$\rho x_2^4x_5^4x_6^4x_7^4x_{10},$$

$$u_1x_1^4x_2x_5^4x_6^4x_7^5x_8x_9x_1x_2^7x_3^5 + u_2x_1^3x_2^2x_5^4x_6^4x_7x_8x_9^2x_1^2x_2^{10}x_3 +$$

$$u_3x_1^2x_2^2x_5^5x_6^5x_7^3x_8x_9^2x_1^8x_2^2x_3^3 + u_4x_1^2x_2^3x_5^4x_6^4x_7^3x_8^2x_9^2x_1^3x_3^{10} +$$

$$u_5x_1^3x_2x_5^4x_6^4x_7^5x_8x_9^2x_1^5x_2^6x_3^2 + u_6x_1^3x_2x_5^5x_6^5x_7^4x_8x_9x_2^4x_3^9 +$$

$$u_7x_1^2x_2^2x_5^5x_6^5x_7^3x_8x_9^2x_1^4x_2^3x_3^6 + u_8x_1x_2^3x_5^4x_6^4x_7^3x_8^2x_9^3x_1^{12}x_2 =$$

$$\rho x_2^5x_5^5x_6^5x_7^5x_{10},$$

$$u_1x_1^4x_2^2x_5^5x_6^5x_7^5x_8x_9^2x_1^2x_2^{10}x_3 + u_2x_1^3x_2^3x_5^5x_6^4x_7^2x_8^2x_9^3x_1^{10}x_3^2 +$$

$$u_3x_1^3x_2^2x_5^6x_6^6x_7^4x_8x_9^2x_1^4x_2^3x_3^6 + u_4x_1^2x_2^3x_5^5x_6^5x_7^4x_8^2x_9^3x_1^{12}x_2 +$$

$$u_5x_1^4x_2x_5^5x_6^6x_7^6x_8x_9^2x_1x_2^7x_3^5 + u_6x_1^3x_2^2x_5^5x_6^5x_7^5x_8^2x_9^2x_1^5x_2^6x_3^2 +$$

$$u_7x_1^3x_2^2x_5^6x_6^6x_7^4x_8x_9^2x_2^4x_3^9 + u_8x_1^2x_2^3x_5^5x_6^4x_7^3x_8^2x_9^3x_1^8x_2^2x_3^3 =$$

$$\rho x_2^6 x_5^6 x_6^6 x_{10},$$

$$u_1 x_1^4 x_2^3 x_5^6 x_6^6 x_7^5 x_8^2 x_9^2 x_1^3 x_3^{10} + u_2 x_1^3 x_2^3 x_5^6 x_6^6 x_7^5 x_8^2 x_9^3 x_1^{12} x_2 +$$

$$u_3 x_1^4 x_2^2 x_5^7 x_6^7 x_7^5 x_8^2 x_9^4 x_3^9 + u_4 x_1^3 x_2^3 x_5^6 x_6^6 x_7^5 x_8^2 x_9^3 x_1^8 x_2^2 x_3^3 +$$

$$u_5 x_1^4 x_2^2 x_5^6 x_6^6 x_7^6 x_8^3 x_9^2 x_1^2 x_2^{10} x_3 + u_6 x_1^4 x_2^2 x_5^6 x_6^6 x_7^6 x_8^2 x_9^2 x_1^7 x_2^5 +$$

$$u_7 x_1^3 x_2^3 x_5^6 x_6^6 x_7^5 x_8^2 x_9^3 x_1^5 x_2^6 x_3^2 + u_8 x_1^3 x_2^3 x_5^6 x_6^6 x_7^5 x_8^2 x_9^3 x_1^4 x_2^3 x_3^6 =$$

$$\rho x_2^7 x_5^7 x_6^7 x_7^7 x_{10}.$$

After dividing both members of each of the latter six equations by common factors, the use of Cramer's rule yields

$$x_1^{12} x_2 = \frac{\rho x_{10} \Delta_1}{\Delta},$$

$$x_1^8 x_2^2 x_3^3 = \frac{\rho x_{10} \Delta_2}{\Delta},$$

$$x_1^5 x_2^6 x_3^2 = \frac{\rho x_{10} \Delta_3}{\Delta},$$

$$x_1^4 x_2^3 x_3^6 = \frac{\rho x_{10} \Delta_4}{\Delta},$$

$$x_1^3 x_3^{10} = \frac{\rho x_{10} \Delta_5}{\Delta},$$

$$x_1^2 x_2^{10} x_3 = \frac{\rho x_{10} \Delta_6}{\Delta},$$

$$x_1 x_2^7 x_3^5 = \frac{\rho x_{10} \Delta_7}{\Delta},$$

$$x_2^4 x_3^9 = \frac{\rho x_{10} \Delta_8}{\Delta},$$

where Δ and $\Delta_m (m=1,2,\ldots,8)$ are determinants which are functions of $u_m (m=1,2,\ldots,8)$ and $X_m (m=1,2,5,6,7,8,9)$. Also, we have from T' that

$$x_1^{13} = \rho x_1,$$

$$x_1^9 x_2 x_3^3 = \rho x_2,$$

$$x_1^6 x_2^5 x_3^2 = \rho x_3,$$

$$x_1^5 x_2^2 x_3^6 = \rho x_4,$$

$$x_1^3 x_2^9 x_3 = \rho x_5,$$

$$x_1^2 x_2^6 x_3^5 = \rho x_6,$$

$$x_1 x_2^3 x_3^9 = \rho x_7,$$

$$x_2^{13} = \rho x_8,$$

$$x_3^{13} = \rho x_9.$$

Using the first equation in each of the previous groups we have

$$\frac{x_1}{x_2} = \frac{x_1 \Delta}{x_{10} \Delta_1}.$$

Now, using the last equation in each we have

$$\frac{x_2^4}{x_3^4} = \frac{x_{10}\Delta_8}{x_9\Delta} .$$

Finally, using the fifth equation in each we obtain

$$\frac{x_3^9}{x_2^9} = \frac{x_{10}\Delta_5}{x_5\Delta} .$$

Hence

$$\frac{x_3^5}{x_2^5} = \frac{x_2^4}{x_3^4} \cdot \frac{x_3^9}{x_2^9} = \frac{x_{10}^2\Delta_5\Delta_8}{x_5x_9\Delta^2}$$

and

$$\frac{x_2}{x_3} = \frac{x_3^4}{x_2^4} \cdot \frac{x_2^5}{x_3^5} = \frac{x_5x_9^2\Delta^3}{x_{10}^3\Delta_5\Delta_8^2} .$$

From these results we obtain

$$(T'^{-1}) \qquad \frac{x_1}{x_1x_5x_9^2\Delta^4} = \frac{x_2}{x_5x_9^2x_{10}\Delta^3\Delta_1} = \frac{x_3}{x_{10}^4\Delta_1\Delta_5\Delta_8^2} .$$

Therefore, between the surface F and the plane there exists a birational correspondence. Thus F is rational.

The transformation

$$x_1' : x_2' : \cdots : x_9' : x_{10}' = x_1 : x_2 : \cdots : x_9 : Ex_{10},$$

which generates on F the involution that has ϕ for its image, corresponds to the involution in the plane generated by the homography H.

If the linear system (3), (4), ..., (12), or (13) is

used instead of the linear system (2), one will get results which will be similar to those obtained in this chapter. However, the involution on the related rational surface F which has ϕ for its image will correspond to an involution in the plane generated by a homography different from H. For the linear systems (3), (4), ..., (12), and (13) this homography will be respectively

$$x_1' : x_2' : x_3' = x_1 : E^7x_2 : E^2x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^9x_2 : E^{10}x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^{10}x_2 : Ex_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^8x_2 : E^6x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^{11}x_2 : E^5x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^2x_2 : E^8x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^5x_2 : E^7x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^3x_2 : E^{12}x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^4x_2 : E^3x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^6x_2 : E^{11}x_3,$$

$$x_1' : x_2' : x_3' = x_1 : E^{12}x_2 : E^9x_3.$$

The involutions generated by these eleven homographies are different from the one generated by H, although all twelve involutions have the same groups.

CHAPTER VI

CONCLUSION

We have considered the involution I_p of period p in Chapter I, where p is a prime number greater than two. A plane cyclic homography which generates I_p was exhibited. Using this homography we were able to determine the coordinates of the p points which define a group of I_p and the equations of the p linear systems of invariant plane curves of order p. Then using the equation of the invariant system which has no base points, we determined the transformation which relates the curves of this system to the hyperplanes of a space S_r, where $(r + 1)$ is the number of terms in this invariant system. Finally, we determined the equations of $(r - 2)$ hypersurfaces which contain the image surface of I_p and proved that this surface was of order p. The results of this chapter are well known.[1] However, the form of the equations representing the p invariant linear systems, the equations representing the transformation relating one of these systems to the hyperplanes of S_r, and the equations representing the $(r - 2)$ hypersurfaces is original so far as the writer has been able to determine. It might be added that the results of the remaining chapters are also original so

[1] Lucien Godeaux, "Sur les homographies planes cycliques," Memoires de la Societe Royale des Sciences de Liege, 3e Serie, Tome XV (1930), pp. 1-26.

far as the author can determine.

A rational surface F is determined in each of the Chapters II, III, and V which is obtained from an involution of period three, five, and thirteen respectively. In each case this surface has equations made up of the equations of the image surface of the involution involved and an additional equation obtained from the equation of the hypersurfaces which are related projectively to one of the invariant plane linear systems whose curves pass through the vertices of the triangle of reference. Also, the order of F equals the square of the period of the involution involved. This surface is in a space which has one more dimension than the space containing the image surface in each of the three cases. In each case one of the invariant plane linear systems which passes through the vertices of the triangle of reference results in a rational surface on which the transformation generating the involution having the same image surface as the involution generated by the plane cyclic homography corresponds to this homography. For the other invariant linear systems of this type the similar transformation on the related rational surface corresponds to a plane cyclic homography which is different from the one which generates the original involution in the plane. Of special interest, however, is the fact that in each of the three cases all involutions generated by those plane cyclic homographies have the same groups.

The main objective in Chapter IV is to extend the results of another paper[2] to an involution of period thirteen. In this chapter a series of five projections is performed on the image surface of the involution. Each projection is made from a point on the surface which is being projected. It is interesting to note that the order of the surface being projected less the multiplicity of the point of projection for this surface equals the order of the resulting surface which is obtained for each projection.

Finally, it would be desirable to extend the results of Chapter V to the case of the involution I_p of period p where p is any prime number greater than two. At the present time the writer sees three major obstacles to be overcome before this objective can be attained. First, it would be desirable to be able to determine the equations of the surface F for this involution. This would depend upon two things. One would be the determination of (r - 2) independent equations which would represent the image surface of I_p and no other surface. The other would be the determination of the equation of the hypersurfaces of order p which correspond to the particular invariant linear system with base points being used. Second, we would like to know whether the surface F is

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[2]Lucien Godeaux, "Recherches sur les involutions cycliques appartenant a une surface algebrique," Bulletin de l'Academie Royale de Belgique (Classe des Sciences), 5e Serie, Tome XVII (1931), pp. 1356-1364.

irreducible or not. This would lead to an investigation of the order of contact of the above hypersurfaces of order p with the image surface of the involution. The greatest difficulty might be determining this order of contact at the branch points which correspond to the points of coincidence of I_p. Third, we would like to know whether the surface F is rational or not. To show that it is by the method used in Chapter V would involve using the equation of the invariant linear system mentioned previously along with the transformation relating projectively the curves of the invariant linear system which has no base points to the hyperplanes of a space S_r to find a solution for the ratios x_1/x_2 and x_2/x_3 in terms of the X's and known constants. The author has given some thought to these three obstacles and feels that it is not impossible to overcome them.

APPENDIXES

APPENDIX I. THE IMAGE SURFACE OF AN I_5.

Consider the homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3,$$

where E is a primitive fifth root of unity. The curve

$$v_1x_1^5 + v_2x_1^2x_2x_3^2 + v_3x_1x_2^3x_3 + v_4x_2^5 + v_5x_3^5 = 0$$

is invariant under H and has no base points. We relate projectively these curves to the hyperplanes of S_4 by taking

$$(T) \qquad \frac{x_1}{x_1^5} = \frac{x_2}{x_1^2x_2x_3^2} = \frac{x_3}{x_1x_2^3x_3} = \frac{x_4}{x_2^5} = \frac{x_5}{x_3^5}.$$

Eliminating x_m(m=1,2,3) in T we get the surface

$$(\phi) \qquad \begin{Vmatrix} x_1x_5 & x_2 & x_3 \\ x_2^2 & x_3 & x_4 \end{Vmatrix} = 0.$$

Surface ϕ is the image of the involution I_5 of period five generated by H.

Now consider the two hypersurfaces

$$x_2^5 = x_1^2x_4x_5^2,$$

$$x_3^5 = x_1x_4^3x_5.$$

These equations determine a surface in S_4. By actual substitution we see that this surface contains the surface ϕ.

Also, it contains the surface given by the equations

$$\left\| \begin{matrix} X_1X_5 & EX_2 & X_3 \\ X_2^2 & X_3 & EX_4 \end{matrix} \right\| = 0.$$

But, this surface and ϕ are not the same.

It can be shown that the surface determined by the two hypersurfaces of the fifth order contains still another surface distinct from ϕ and the one exhibited above. We conclude from this that the two hypersurfaces contain a surface, other than ϕ, which is reducible.

APPENDIX II. THE IMAGE CURVES OF A LINEAR SYSTEM INVARIANT UNDER A HOMOGRAPHY OF PERIOD THREE

The homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3,$$

where E is a primitive cube root of unity, generates an involution I_3 of period three. The two linear systems of plane curves

$$v_1x_1^3 + v_2x_1x_2x_3 + v_3x_2^3 + v_4x_3^3 = 0$$

and

$$u_1x_1^2x_3 + u_2x_1x_2^2 + u_3x_2x_3^2 = 0$$

are invariant under H. We shall designate the former system by $|C|$ and the latter by $|C_1|$. A curve of $|C|$ will be designated by C and a similar notation will hold for the system $|C_1|$. The curves C have no base points, but the curves C_1 have for base points $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$. We relate projectively the curves C to the planes of a space S_3 of three dimensions by taking

$$(T) \qquad \frac{x_1}{x_1^3} = \frac{x_2}{x_1x_2x_3} = \frac{x_3}{x_2^3} = \frac{x_4}{x_3^3} .$$

Eliminating x_m(m=1,2,3) in T we obtain the surface

$$(\phi) \qquad x_2^3 = x_1x_3x_4.$$

This surface is the image of I_3 and is of order three. Also, we note that the curves C_1 pass simply through O_1, O_2, and O_3.

Each curve C_1 is tangent to the line $x_3 = 0$ at O_1, to the line $x_1 = 0$ at O_2, and to the line $x_2 = 0$ at O_3.

It will be our aim here to study the behavior of the image curves in S_3 of the curves C_1. These image curves form a system of curves which we shall designate by $|A|$. Any one of the curves of $|A|$ will be designated by A. The curves A lie on ϕ. We are especially interested in the behavior of the curves A in the domain of the points $O_1'(1,0,0,0)$, $O_3'(0,0,1,0)$, and $O_4'(0,0,0,1)$, which are the image points of O_1, O_2, and O_3 respectively. In arriving at our desired aim we shall employ the methods used by other authors.[1]

The quadratic transformation

$$x_1 : x_2 : x_3 = y_1^2 : y_1y_2 : y_2y_3$$

makes the point $y_2 = y_3 = 0$ correspond to the point O_{12} which lies on the line $x_3 = 0$ infinitely near O_1. If we apply this transformation to the equation of the curves C which pass through O_1, we obtain

$$(v_2y_3 + v_3y_2)y_1^3 + v_4y_2y_3^3 = 0.$$

Hence, the curves C which pass through O_1 also pass simply

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[1] J. Dessart, "Sur les surfaces representant l'involution engendree par une homographie de periode cinq du plan," Memoires de la Societe Royale des Sciences de Liege, 3e Serie, Tome XVI (1931), pp. 1-23, and W. R. Hutcherson and S. T. Gormsen, "Maps of Certain Algebraic Curves Invariant under Cyclic Involutions of Periods Three, Five, and Seven," Canadian Journal of Mathematics, Vol. VI (1954), pp. 92-98.

through O_{12}. From T we see that

$$\frac{X_2}{y_1^3 y_3} = \frac{X_3}{y_1^3 y_2} = \frac{X_4}{y_2 y_3^3}.$$

By placing $y_3 = ky_2$ and letting y_2 tend to zero we obtain the parametric equations

$$\frac{X_2}{k} = \frac{X_3}{1}, \quad X_4 = 0.$$

This is one of the tangent planes to surface ϕ at O_1'. Now, applying this quadratic transformation to the equation of curves C_1 we obtain

$$(u_1 y_3 + u_2 y_2) y_1^3 + u_3 y_2^2 y_3^2 = 0.$$

Again, placing $y_3 = ky_2$ and letting y_2 tend to zero we have

$$u_1 k + u_2 = 0.$$

Solving this equation for k and substituting in the previous equation we get

$$u_1 X_2 + u_2 X_3 = 0, \quad X_4 = 0.$$

This is the tangent element to a curve A at O_1'. Since this tangent element is a line, we see that the curves A pass simply through O_1'.

We could determine the tangent elements to the curves A at O_3' and O_4' in a similar manner using other quadratic transformations. But these results can be more easily obtained by noting that the equations of the curves C and C_1

remain unchanged in applying the permutation $p_1 = (123)$ to the subscripts of the x's and u's and the permutation $P_1 = (134)$ to the subscripts of the v's. Hence, applying p_1 to the subscripts of u and P_1 to the subscripts of X we see that the tangent element to a curve A at O_3' is

$$u_2X_2 + u_3X_4 = 0, \quad X_1 = 0.$$

Therefore, the curves A pass simply through O_3'.

Using the permutations $p_2 = (132)$ and $P_2 = (143)$ in a similar manner we obtain the tangent element

$$u_3X_2 + u_1X_1 = 0, \quad X_3 = 0$$

to a curve A at O_4'. Hence, the curves A also pass simply through O_4'.[2]

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[2] In this appendix we have extended some of the results found in Lucien Godeaux, Cours de Geometrie Superieure, Fascicule II (Liege: Sciences et Lettres, 1937), pp. 21-26.

APPENDIX III. THE IMAGE CURVES OF A LINEAR SYSTEM INVARIANT UNDER A HOMOGRAPHY OF PERIOD THIRTEEN

The homography

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^4 x_3,$$

where E is a primitive thirteenth root of unity, generates an involution I_{13} of period thirteen. The two linear systems of plane curves

$$v_1 x_1^{13} + v_2 x_1^9 x_2 x_3^3 + v_3 x_1^6 x_2^5 x_3^2 + v_4 x_1^5 x_2^2 x_3^6 + v_5 x_1^3 x_2^9 x_3 +$$

$$v_6 x_1^2 x_2^6 x_3^5 + v_7 x_1 x_2^3 x_3^9 + v_8 x_2^{13} + v_9 x_3^{13} = 0$$

and

$$u_1 x_1^{12} x_2 + u_2 x_1^8 x_2^2 x_3^3 + u_3 x_1^5 x_2^6 x_3^2 + u_4 x_1^4 x_2^3 x_3^6 + u_5 x_1^3 x_3^{10} +$$

$$u_6 x_1^2 x_2^{10} x_3 + u_7 x_1 x_2^7 x_3^5 + u_8 x_2^4 x_3^9 = 0$$

are invariant under H. The former system will be designated by $|C|$ and the latter by $|C_1|$. A curve of $|C|$ and $|C_1|$ will be designated by C and C_1 respectively. The curves C have no base points, but the curves C_1 have for base points $O_1(1,0,0)$, $O_2(0,1,0)$, and $O_3(0,0,1)$. Also, we note that the curves C_1 pass simply through O_1, triply through O_2, and triply through O_3. Each curve C_1 is tangent to the line $x_2 = 0$ at O_1, to the lines $x_1 = 0$ and $x_3 = 0$ at O_2, and to the line $x_1 = 0$ at O_3.

We relate projectively the curves C to the hyperplanes

of S_8 by taking

$$(T) \qquad \frac{x_1}{x_1^{13}} = \frac{x_2}{x_1^9 x_2 x_3^3} = \frac{x_3}{x_1^6 x_2^5 x_3^2} = \frac{x_4}{x_1^5 x_2^2 x_3^6} = \frac{x_5}{x_1^3 x_2^9 x_3} =$$

$$\frac{x_6}{x_1^2 x_2^6 x_3^5} = \frac{x_7}{x_1 x_2^3 x_3^9} = \frac{x_8}{x_2^{13}} = \frac{x_9}{x_3^{13}} .$$

Eliminating $x_m (m=1,2,3)$ in T we obtain a surface ϕ given by the equations

$$(\phi) \qquad \left\| \begin{matrix} x_1x_6 & x_2 & x_3 & x_4 & x_5 & x_3x_7 & x_4x_9 \\ x_3^2 & x_3 & x_5 & x_6 & x_8 & x_6^2 & x_7^2 \end{matrix} \right\| = 0.$$

This surface is the image of I_{13} and is of order thirteen.

The curves C_1 are imaged in S_8 by curves which lie on ϕ. We shall designate this system of image curves by $|A|$ and any one of them by A. Of special interest is the behavior of the curves A in the domain of the points $O_1'(1,0,0,0,0,0,0,0,0)$, $O_8'(0,0,0,0,0,0,0,0,1,0)$, and $O_9'(0,0,0,0,0,0,0,0,0,1)$. We shall employ the methods described in Appendix II for this investigation.

The quadratic transformation

$$x_1 : x_2 : x_3 = y_1^2 : y_2y_3 : y_1y_3$$

makes the point $y_2 = y_3 = 0$ correspond to the point O_{13} which lies on the line $x_2 = 0$ infinitely near O_1. If we apply this transformation nine times in succession to the

equation of the curves C_1, we obtain

$$(u_1y_2 + u_5y_3)y_1^{91} + u_2y_1^{78}y_2^{2}y_3^{12} + u_3y_1^{39}y_2^{6}y_3^{47} + u_4y_1^{65}y_2^{3}y_3^{24} +$$

$$u_6y_2^{10}y_3^{82} + u_7y_1^{26}y_2^{7}y_3^{59} + u_8y_1^{52}y_2^{4}y_3^{36} = 0.$$

Hence, the curves C_1 pass simply through O_1, O_{13}, ..., and $O_{1333333333}$. Placing $y_3 = ky_2$ in this equation, we get

$$(u_1 + u_5k)y_1^{91} + k^{12}u_2y_1^{78}y_2^{13} + k^{47}u_3y_1^{39}y_2^{52} + k^{24}u_4y_1^{65}y_2^{26} +$$

$$k^{82}u_6y_2^{91} + k^{59}u_7y_1^{26}y_2^{65} + k^{36}u_8y_1^{52}y_2^{39} = 0.$$

Now, letting y_2 tend to zero we have

$$u_1 + u_5k = 0.$$

Solving for k we get

$$k = -\frac{u_1}{u_5}.$$

If we now apply the previous quadratic transformation to the equation of the curves C which pass through O_1, we obtain

$$v_2y_1^{21}y_2 + v_3y_1^{14}y_2^{5}y_3^{3} + v_4y_1^{16}y_2^{2}y_3^{4} + v_5y_1^{7}y_2^{9}y_3^{6} + v_6y_1^{9}y_2^{6}y_3^{7} +$$

$$v_7y_1^{11}y_2^{3}y_3^{8} + v_8y_2^{13}y_3^{9} + v_9y_1^{13}y_3^{9} = 0.$$

Applying the same transformation to this equation eight times in succession yields

$$(v_2y_2 + v_9y_3)y_1^{117} + v_3y_1^{78}y_2^{5}y_3^{35} + v_4y_1^{104}y_2^{2}y_3^{12} +$$

$$v_5 y_1^{39} y_2^9 y_3^{70} + v_6 y_1^{65} y_2^6 y_3^{47} + v_7 y_1^{91} y_2^3 y_3^{24} + v_8 y_2^{13} y_3^{105} = 0.$$

Hence, the curves C which pass through O_1 have a 4-tuple point O_1 and nine simple points O_{13}, O_{133}, ..., and $O_{1333333333}$. From T and the above equation we have

$$\frac{X_2}{y_1^{117} y_2} = \frac{X_3}{y_1^{78} y_2^5 y_3^{35}} = \frac{X_4}{y_1^{104} y_2^2 y_3^{12}} = \frac{X_5}{y_1^{39} y_2^9 y_3^{70}} =$$

$$\frac{X_6}{y_1^{65} y_2^6 y_3^{47}} = \frac{X_7}{y_1^{91} y_2^3 y_3^{24}} = \frac{X_8}{y_2^{13} y_3^{105}} = \frac{X_9}{y_1^{117} y_3} .$$

Placing $y_3 = ky_2$ this reduces to

$$\frac{X_2}{y_1^{117}} = \frac{X_3}{k^{35} y_1^{78} y_2^{39}} = \frac{X_4}{k^{12} y_1^{104} y_2^{13}} = \frac{X_5}{k^{70} y_1^{39} y_2^{78}} = \frac{X_6}{k^{47} y_1^{65} y_2^{52}} =$$

$$\frac{X_7}{k^{24} y_1^{91} y_2^{26}} = \frac{X_8}{k^{105} y_2^{117}} = \frac{X_9}{k y_1^{117}} .$$

Now, letting y_2 tend to zero we get the parametric equations

$$\frac{X_2}{1} = \frac{X_9}{k} , \qquad X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0.$$

These are the equations of a plane which is tangent to the surface ϕ at O_1' since this point is the image of O_1.

We obtain the tangent element to the curves A at O_1' by substituting the value of k which we have previously found in the equations of this tangent plane. This results

in the equations

$$u_1X_2 + u_5X_9 = 0, \quad X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0.$$

Since this tangent element is a line, we conclude that the curves A pass simply through O_1'.

We shall now utilize the two quadratic transformations

$$x_1 : x_2 : x_3 = y_1y_3 : y_2^2 : y_2y_3,$$

$$x_1 : x_2 : x_3 = y_1y_2 : y_2^2 : y_1y_3.$$

The first makes the point $y_1 = y_3 = 0$ correspond to the point O_{23} which lies on the line $x_1 = 0$ infinitely near O_2. The latter makes this point correspond to O_{21} which lies on the line $x_3 = 0$ infinitely near O_2. If we apply the first transformation three times in succession to the equation of the curves C_1 and the latter nine times in succession to this equation, we obtain respectively the equations

$$(u_6y_1^2 + u_7y_1y_3 + u_8y_3^2)y_2^{39} + (u_3y_1^5y_3^{10} + u_4y_1^4y_3^{11} + u_5y_1^3y_3^{12})y_2^{26} + u_2y_1^8y_2^{13}y_3^{20} + u_1y_1^{12}y_3^{29} = 0$$

and

$$(u_1y_1 + u_6y_3)y_2^{91} + u_2y_1^{24}y_2^{65}y_3^3 + u_3y_1^{12}y_2^{78}y_3^2 + u_4y_1^{47}y_2^{39}y_3^6 + u_5y_1^{82}y_3^{10} + u_7y_1^{35}y_2^{52}y_3^5 + u_8y_1^{70}y_2^{13}y_3^9 = 0.$$

Hence, the curves C_1 have a triple point O_2; three double points O_{23}, O_{233}, and O_{2333}; and nine simple points O_{21}, O_{211},

..., and $O_{2111111111}$. If we place $y_1 = ky_3$ in each of these equations and then let y_3 tend to zero, we obtain respectively

$$u_6k^2 + u_7k + u_8 = 0$$

and

$$u_1k + u_6 = 0.$$

If we apply the first of these quadratic transformations three times in succession to the equation of the curves C which pass through O_2 and the latter nine times in succession to this equation, we obtain respectively the equations

$$(v_5y_1^3 + v_6y_1^2y_3 + v_7y_1y_3^2 + v_9y_3^3)y_2^{39} + (v_3y_1^6y_3^{10} +$$

$$v_4y_1^5y_3^{11})y_2^{26} + v_2y_1^9y_2^{13}y_3^{20} + v_1y_1^{13}y_3^{29} = 0$$

and

$$(v_1y_1 + v_5y_3)y_2^{117} + v_3y_1^{12}y_2^{104}y_3^2 + v_2y_1^{24}y_2^{91}y_3^3 +$$

$$v_6y_1^{35}y_2^{78}y_3^5 + v_4y_1^{47}y_2^{65}y_3^6 + v_7y_1^{70}y_2^{39}y_3^9 + v_9y_1^{105}y_3^{13} = 0.$$

Hence, the curves C which pass through O_2 have a 4-tuple point O_2; three triple points O_{23}, O_{233}, and O_{2333}; and nine simple points O_{21}, O_{211}, ..., and $O_{2111111111}$. From T and these equations we obtain respectively

$$\frac{x_1}{y_1^{13}y_3^{29}} = \frac{x_2}{y_1^9y_2^{13}y_3^{20}} = \frac{x_3}{y_1^6y_2^{26}y_3^{10}} = \frac{x_4}{y_1^5y_2^{26}y_3^{11}} = \frac{x_5}{y_1^3y_2^{39}} =$$

$$\frac{x_6}{y_1^2 y_2^{39} y_3} = \frac{x_7}{y_1 y_2^{39} y_3^2} = \frac{x_9}{y_2^{39} y_3^3}$$

and

$$\frac{x_1}{y_1 y_2^{117}} = \frac{x_2}{y_1^{24} y_2^{91} y_3^3} = \frac{x_3}{y_1^{12} y_2^{104} y_3^2} = \frac{x_4}{y_1^{47} y_2^{65} y_3^6} = \frac{x_5}{y_2^{117} y_3} =$$

$$\frac{x_6}{y_1^{35} y_2^{78} y_3^5} = \frac{x_7}{y_1^{70} y_2^{39} y_3^9} = \frac{x_9}{y_1^{105} y_3^{13}} .$$

If we substitute $y_1 = ky_3$ in each of these relations and then let y_3 tend to zero, we have respectively the parametric equations

$$\frac{x_5}{k^3} = \frac{x_6}{k^2} = \frac{x_7}{k} = \frac{x_9}{1}, \quad x_1 = x_2 = x_3 = x_4 = 0$$

and

$$\frac{x_1}{k} = \frac{x_5}{1}, \qquad x_2 = x_3 = x_4 = x_6 = x_7 = x_9 = 0.$$

Since O_8' is the image of O_2, these are the equations of a cubic cone and a plane which are tangent to the surface ϕ at O_8'.

In order to get the tangent elements to the curves A at O_8', we substitute the values of k from

$$u_6 k^2 + u_7 k + u_8 = 0$$

in

$$\frac{X_5}{k^3} = \frac{X_6}{k^2} = \frac{X_7}{k} = \frac{X_9}{1}, \qquad X_1 = X_2 = X_3 = X_4 = 0$$

and the value of k from

$$u_1 k + u_6 = 0$$

in

$$\frac{X_1}{k} = \frac{X_5}{1}, \quad X_2 = X_3 = X_4 = X_6 = X_7 = X_9 = 0.$$

Hence, the tangent elements to the curves A at O_8' are composed of three lines and consequently these curves have a triple point at this point.

Finally, we shall utilize the two quadratic transformations

$$x_1 : x_2 : x_3 = y_1 y_2 : y_2 y_3 : y_3^2,$$

$$x_1 : x_2 : x_3 = y_1 y_3 : y_1 y_2 : y_3^2.$$

The former makes the point $y_1 = y_2 = 0$ correspond to the point O_{32} which lies on the line $x_1 = 0$ infinitely near O_3. The latter makes this point correspond to the point O_{31} which lies on the line $x_2 = 0$ infinitely near O_3. If we apply the first of these transformations once to the equation of the curves C_1, we obtain

$$u_1 y_1^{12} y_2^{10} + u_2 y_1^8 y_2^7 y_3^7 + u_3 y_1^5 y_2^8 y_3^9 + u_4 y_1^4 y_2^4 y_3^{14} + u_5 y_1^3 y_3^{19} +$$

$$u_6 y_1^2 y_2^9 y_3^{11} + u_7 y_1 y_2^5 y_3^{16} + u_8 y_2 y_3^{21} = 0.$$

Applying the second of these transformations two times in succession to the above equation yields

$$(u_5y_1 + u_8y_2)y_3^{39} + (u_4y_1^{10}y_2^4 + u_7y_1^9y_2^5)y_3^{26} + (u_2y_1^{20}y_2^7 +$$

$$u_3y_1^{19}y_2^8 + u_6y_1^{18}y_2^9)y_3^{13} + u_1y_1^{30}y_2^{10} = 0.$$

Hence, the curves C_1 have a triple point O_3 and three simple points O_{32}, O_{321}, and O_{3211}. If we place $y_2 = ky_1$ in this equation and then let y_1 tend to zero, we obtain

$$u_5 + u_8k = 0.$$

By utilizing these quadratic transformations we find that the curves C which pass through O_3 do not pass through the point O_{3211}. Therefore, we shall investigate the curves C which pass through O_3 with distinct tangent lines $x_1 = 0$ and $x_2 = 0$ at this point. The equation of these curves is obtained from the equation of the curves C by taking $v_7 = v_9 = 0$. If we now apply the first of these quadratic transformations once to this equation, we obtain

$$v_1y_1^{13}y_2^6 + v_2y_1^9y_2^3y_3^7 + v_3y_1^6y_2^4y_3^9 + v_4y_1^5y_3^{14} + v_5y_1^3y_2^5y_3^{11} +$$

$$v_6y_1^2y_2y_3^{16} + v_8y_2^6y_3^{13} = 0.$$

Applying now the second quadratic transformation twice in succession yields

$$v_1 y_1^{16} y_2^{6} + v_2 y_1^{9} y_2^{3} y_3^{10} + v_3 y_1^{7} y_2^{4} y_3^{11} + v_4 y_1^{2} y_3^{20} + v_5 y_1^{5} y_2^{5} y_3^{12} +$$

$$v_6 y_2 y_3^{21} + v_8 y_1^{3} y_2^{6} y_3^{13} = 0$$

and

$$(v_4 y_1 + v_6 y_2) y_3^{26} + (v_2 y_1^{11} y_2^{3} + v_3 y_1^{10} y_2^{4} + v_5 y_1^{9} y_2^{5} +$$

$$v_8 y_1^{8} y_2^{6}) y_3^{13} + v_1 y_1^{21} y_2^{6} = 0.$$

Hence, the curves C which pass through O_3 with distinct tangent lines $x_1 = 0$ and $x_2 = 0$ at this point have a 7-tuple point O_3, a triple point O_{32}, and two simple points O_{321} and O_{3211}. From T and the above equation we have

$$\frac{x_1}{y_1^{21} y_2^{6}} = \frac{x_2}{y_1^{11} y_2^{3} y_3^{13}} = \frac{x_3}{y_1^{10} y_2^{4} y_3^{13}} = \frac{x_4}{y_1 y_3^{26}} = \frac{x_5}{y_1^{9} y_2^{5} y_3^{13}} =$$

$$\frac{x_6}{y_2 y_3^{26}} = \frac{x_8}{y_1^{8} y_2^{6} y_3^{13}}.$$

Substituting $y_2 = k y_1$ in these relations and letting y_1 tend to zero, we get the parametric equations

$$\frac{x_4}{1} = \frac{x_6}{k}, \quad x_1 = x_2 = x_3 = x_5 = x_8 = 0.$$

If the surface ϕ is projected from O_9' on the hyperplane $x_9 = 0$, the projection will be a surface ϕ' which lies in the space $x_9 = 0$. The above equations represent a tangent

plane to the surface ϕ' at O_7' in this space. The projections of the curves A from O_9' will lie on ϕ'. Hence, the value of k obtained previously substituted in the above equations gives the tangent element at O_7' to the projections of the curves A which lie on ϕ'. This tangent consequently has in S_8 the equations

$$u_5X_4 + u_8X_6 = 0, \quad X_1 = X_2 = X_3 = X_5 = X_8 = X_9 = 0.$$

Since this tangent is a line, we see that the point O_7' is simple for the projections of the curves A from O_9'.

We are now able to infer from these results that the curves A pass simply through O_9'. The projections of these curves from O_9' are cut out on ϕ' by the quadratic

$$u_1X_1X_8 + u_2X_2X_8 + u_3X_3X_8 + u_4X_4X_8 + u_5X_5X_7 + u_6X_5X_8 +$$

$$u_7X_6X_8 + u_8X_7X_8 = 0.$$

This equation is obtained by multiplying both members of the equation of the curves C_1 by $x_1x_2^{12}$ and applying T. The equations of the curves A are

$$\left\| \begin{matrix} X_1X_6 & X_2 & X_3 & X_4 & X_5 & X_3X_7 & X_4X_9 & f(X_1,\ldots,X_7) \\ X_3^2 & X_3 & X_5 & X_6 & X_8 & X_8^2 & X_7^2 & -u_5X_7 \end{matrix} \right\| = 0,$$

where

$$f(X_1,\ldots,X_7) \equiv u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4 + u_6X_5 + u_7X_6 + u_8X_7.$$

The line which passes through the points $0_7'$ and $0_9'$ has the equation

$$X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_8 = 0.$$

This line lies in the above quadratic and intersects the curves A only at $0_9'$. Hence, the neighborhood of $0_9'$ on these curves projects into $0_7'$. We conclude from this that the point $0_9'$ is simple for the curves A and that the tangent line to these curves at this point is the line which passes through $0_7'$ and $0_9'$. Further investigation will show that the point $0_9'$ is a 4-tuple point for the surface ϕ with the tangent cone decomposing into a cubic cone and a plane. It can also be shown that the tangent line to the curves A at $0_9'$ lies in both this cubic cone and plane.

BIBLIOGRAPHY

Busemann, Herbert. Introduction to Algebraic Manifolds. Princeton: Princeton University Press, 1939.

Cartan, Elie J. Lecons sur la Geometrie Projective Complexe. Paris: Gauthier-Villars, 1950.

Dessart, J. "Sur les surfaces representant l'involution engendree par une homographie de periode cinq du plan," Memoires de la Societe Royale des Sciences de Liege, 3e Serie, Tome XVI (1931), pp. 1-23.

Du Val, Patrick. "Algebraic Loci Whose Curve Sections Are Hyperelliptic," The Journal of the London Mathematical Society, Vol. 28 (1953), pp. 1-8.

Forsyth, Andrew R. Geometry of Four Dimensions. 2 vols. Cambridge: The University Press, 1930.

Godeaux, Lucien. Cours de Geometrie Superieure. Fascicule II. Liege: Librairie Bourguignon, 1937.

_____. "Etude elementaire sur l'homographie plane de periode trois et sur une surface cubique," Nouvelles Annales de Mathematiques, 4e Serie, Tome XVI (1916), pp. 49-61.

_____. Geometrie Algebrique. 2 tomes. Liege: Sciences et Lettres, 1948-1949.

_____. Introduction a la Geometrie Projective Hyperspatiale. Liege: Librairie Bourguignon, 1939.

_____. Introduction a la Geometrie Superieure. 2e edition. Liege: G. Thone, 1946.

_____. Memoire sur les Surfaces Multiples. Liege: G. Thone, 1952.

_____. "Recherches sur les involutions cycliques appartenant a une surface algebrique," Bulletin de l'Academie Royale de Belgique (Classe de Sciences), 5e Serie, Tome XVII (1931), pp. 1356-1364.

_____. "Sur les homographies planes cycliques," Memoires de la Societe Royale des Sciences de Liege, 3e Serie, Tome XV (1930), pp. 1-26.

Gormsen, Svend T. "Maps of Certain Algebraic Curves Invariant under Cyclic Involutions of Periods Three, Five, and Seven." Unpublished form of Ph. D. dissertation, Department of Mathematics, University of Florida, 1953.

Graustein, William C. Introduction to Higher Geometry. New York: The Macmillan Company, 1946.

Hudson, Hilda P. Cremona Transformations in Plane and Space. Cambridge: The University Press, 1927.

Hutcherson, W. R. "A Cyclic Involution of Order Seven," Bulletin of the American Mathematical Society, Vol. 40 (1934), pp. 143-151.

_____. "A Cyclic Involution of Period Eleven," Canadian Journal of Mathematics, Vol. III (1951), pp. 155-158.

_____. "Maps of Certain Cyclic Involutions on Two Dimensional Carriers," Bulletin of the American Mathematical Society, Vol. 37 (1931), pp. 759-765.

_____. "Invariant Curves of Order Eight," Revista Matematica Y Fisica Teorica, Serie A, Vol. 9 (1952), pp. 13-14.

_____, et Childress, N. A. "Etude d'une involution cyclique de periode cinq," Bulletin de l'Academie Royale de Belgique (Classe de Sciences), 6e Serie, Tome XL (1954), pp. 103-108.

_____, and Gormsen, S. T. "Maps of Certain Algebraic Curves Invariant under Cyclic Involutions of Periods Three, Five, and Seven," Canadian Journal of Mathematics, Vol. VI (1954), pp. 92-98.

Manning, Henry P. Geometry of Four Dimensions. New York: The Macmillan Company, 1914.

Morelock, James C. "Invariants with Respect to Special Projective Transformations." Unpublished form of Ph. D. dissertation, Department of Mathematics, University of Florida, 1952.

Sommerville, D. M. Y. Analytical Geometry of Three Dimensions. Cambridge: The University Press, 1947.

Veblen, Oswald, and Young, John W. Projective Geometry. 2 vols. Boston: Ginn and Company, 1910-1918.

Winger, R. M. An Introduction to Projective Geometry. Boston: D. C. Heath and Company, 1923.

Woods, Frederick S. Higher Geometry. Boston: Ginn and Company, 1922.

BIOGRAPHICAL SKETCH

Noel Arthur Childress was born in Lafayette County, Mississippi, on January 12, 1920. The early years of his life were spent in this locality. He attended the public schools of this county and graduated from University High School, Oxford, Mississippi, in 1937. Immediately thereafter he entered the University of Mississippi and received the degree of Bachelor of Arts in Education with a major in mathematics from this institution in 1941. He entered the University of Mississippi again in 1946 for further study on the graduate level and received the Master of Arts degree with a major in mathematics from this institution in 1947. He pursued graduate study at the University of Tennessee for one year during 1947 and 1948. After attending the University of Florida for the summer sessions of 1950 and 1951 he entered this institution in June, 1952, and has pursued graduate study continuously since that time.

The author taught in the high schools of Mississippi for one and one-half years during 1941 and 1942. He spent four years as an instructor of mathematics at the University of Mississippi during 1948 to 1952. While taking advanced work he has been a graduate assistant at all three institutions of higher learning which he has attended.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

June 7, 1954

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